35. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	…
M	M	…
m_1	m_2	Coefficients
m_1	m_2	
.	.	
.	.	
.	.	

1/2×1/2

	$+1$	
$+1/2\ +1/2$	1	0
	0	0

$+1/2\ -1/2$	1/2	1/2	1
$-1/2\ +1/2$	1/2	−1/2	−1

$-1/2\ -1/2$	1	

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

	$+5/2$	
$+2\ +1/2$	1	$5/2\ \ 3/2$
	$+3/2\ +3/2$	

$+2\ -1/2$	1/5	4/5	5/2 3/2
$+1\ +1/2$	4/5	−1/5	+1/2 +1/2

$+1\ -1/2$	2/5	3/5	5/2	3/2
$0\ +1/2$	3/5	−2/5	−1/2	−1/2

$0\ -1/2$	3/5	2/5	5/2	3/2	
$-1\ +1/2$	2/5	−3/5	−3/2	−3/2	

$-1\ -1/2$	4/5	1/5	5/2			
$-2\ +1/2$	1/5	−4/5	−5/2			

$-2\ -1/2$	1						

1×1/2

	$+3/2$	
$+1\ +1/2$	1	$3/2\ \ 1/2$
		$+1/2\ +1/2$

$+1\ -1/2$	1/3	2/3	3/2 1/2
$0\ +1/2$	2/3	−1/3	−1/2 −1/2

$0\ -1/2$	2/3	1/3	3/2	
$-1\ +1/2$	1/3	−2/3	−3/2	

$-1\ -1/2$	1				

3/2×1/2

	$+2$		
$+3/2\ +1/2$	1	2	1
	$+1$	$+1$	

$+3/2\ -1/2$	1/4	3/4	2	1
$+1/2\ +1/2$	3/4	−1/4	0	0

$+1/2\ -1/2$	1/2	1/2	2	1	
$-1/2\ +1/2$	1/2	−1/2	−1	−1	

$-1/2\ -1/2$	3/4	1/4	2			
$-3/2\ +1/2$	1/4	−3/4	−2			

$-3/2\ -1/2$	1						

2×1

	$+3$	
$+2\ +1$	1	$+2\ \ +2$

$+2\ \ 0$	1/3	2/3	3 2 1
$+1\ +1$	2/3	−1/3	+1 +1 +1

$+2\ -1$	1/15	1/3	3/5	
$+1\ \ 0$	8/15	1/6	−3/10	
$0\ +1$	2/5	−1/2	1/10	

$+1\ -1$	1/5	1/2	3/10			
$0\ \ 0$	3/5	0	−2/5			
$-1\ +1$	1/5	−1/2	3/10			

$0\ -1$	2/5	1/2	1/10				
$-1\ \ 0$	8/15	−1/6	−3/10				
$-2\ +1$	1/15	−1/3	3/5				

$-1\ -1$	2/3	1/3	3					
$-2\ \ 0$	1/3	−2/3	−3					

$-2\ -1$	1								

3/2×1

	$+5/2$	
$+3/2\ +1$	1	$5/2\ \ 3/2$
		$+3/2\ +3/2$

$+3/2\ \ 0$	2/5	3/5	5/2	3/2	1/2
$+1/2\ +1$	3/5	−2/5	+1/2	+1/2	+1/2

$+3/2\ -1$	1/10	2/5	1/2			
$+1/2\ \ 0$	3/5	1/15	−1/3			
$-1/2\ +1$	3/10	−8/15	1/6			

$+1/2\ -1$	3/10	8/15	1/6					
$-1/2\ \ 0$	3/5	−1/15	−1/3					
$-3/2\ +1$	1/10	−2/5	1/2					

$-1/2\ -1$	3/5	2/5	5/2						
$-3/2\ \ 0$	2/5	−3/5	−5/2						

$-3/2\ -1$	1									

1×1

	$+2$	
$+1\ +1$	1	$+1\ \ +1$

$+1\ \ 0$	1/2	1/2	2 1 0
$0\ +1$	1/2	−1/2	0 0 0

$+1\ -1$	1/6	1/2	1/3	
$0\ \ 0$	2/3	0	−1/3	
$-1\ +1$	1/6	−1/2	1/3	

$0\ -1$	1/2	1/2	2		
$-1\ \ 0$	1/2	−1/2	−2		

$-1\ -1$	1					

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^{\ell}_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{j}_{m',m} = (-1)^{m-m'} d^{j}_{m,m'} = d^{j}_{-m,-m'}$$

3/2×3/2

	$+3$	
$+3/2\ +3/2$	1	$3\ \ 2$
		$+2\ \ +2$

$+3/2\ +1/2$	1/2	1/2	3 2 1
$+1/2\ +3/2$	1/2	−1/2	+1 +1 +1

$+3/2\ -1/2$	1/5	1/2	3/10	
$+1/2\ +1/2$	3/5	0	−2/5	
$-1/2\ +3/2$	1/5	−1/2	3/10	

$+3/2\ -3/2$	1/20	1/4	9/20	1/4		
$+1/2\ -1/2$	9/20	1/4	−1/20	−1/4		
$-1/2\ +1/2$	9/20	−1/4	−1/20	1/4		
$-3/2\ +3/2$	1/20	−1/4	9/20	−1/4		

$+1/2\ -3/2$	1/5	1/2	3/10				
$-1/2\ -1/2$	3/5	0	−2/5				
$-3/2\ +1/2$	1/5	−1/2	3/10				

$-1/2\ -3/2$	1/2	1/2	3					
$-3/2\ -1/2$	1/2	−1/2	−3					

$-3/2\ -3/2$	1								

2×3/2

	$+7/2$	
$+2\ +3/2$	1	$7/2\ \ 5/2$
		$+5/2\ +5/2$

$+2\ +1/2$	3/7	4/7	7/2 5/2 3/2
$+1\ +3/2$	4/7	−3/7	+3/2 +3/2 +3/2

$+2\ -1/2$	1/7	16/35	2/5	
$+1\ +1/2$	4/7	1/35	−2/5	
$0\ +3/2$	2/7	−18/35	1/5	

$+2\ -3/2$	1/35	6/35	2/5	2/5				
$+1\ -1/2$	12/35	5/14	0	−3/10				
$0\ +1/2$	18/35	−3/35	−1/5	1/5				
$-1\ +3/2$	4/35	−27/70	2/5	−1/10				

$+1\ -3/2$	4/35	27/70	2/5	1/10						
$0\ -1/2$	18/35	3/35	−1/5	−1/5						
$-1\ +1/2$	12/35	−5/14	0	3/10						
$-2\ +3/2$	1/35	−6/35	2/5	−2/5						

$0\ -3/2$	2/7	18/35	1/5								
$-1\ -1/2$	4/7	−1/35	−2/5								
$-2\ +1/2$	1/7	−16/35	2/5								

$-1\ -3/2$	4/7	3/7	7/2									
$-2\ -1/2$	3/7	−4/7	−7/2									

$-2\ -3/2$	1												

2×2

	$+4$	
$+2\ +2$	1	$4\ \ 3$
		$+3\ \ +3$

$+2\ +1$	1/2	1/2	4 3 2
$+1\ +2$	1/2	−1/2	+2 +2 +2

$+2\ \ 0$	3/14	1/2	2/7	
$+1\ +1$	4/7	0	−3/7	
$0\ +2$	3/14	−1/2	2/7	

$+2\ -1$	1/14	3/10	3/7	1/5	
$+1\ \ 0$	3/7	1/5	−1/14	−3/10	
$0\ +1$	3/7	−1/5	−1/14	3/10	
$-1\ +2$	1/14	−3/10	3/7	−1/5	

$+2\ -2$	1/70	1/10	2/7	2/5	1/5				
$+1\ -1$	8/35	2/5	1/14	−1/10	−1/5				
$0\ \ 0$	18/35	0	−2/7	0	1/5				
$-1\ +1$	8/35	−2/5	1/14	1/10	−1/5				
$-2\ +2$	1/70	−1/10	2/7	−2/5	1/5				

$+1\ -2$	1/14	3/10	3/7	1/5						
$0\ -1$	3/7	1/5	−1/14	−3/10						
$-1\ \ 0$	3/7	−1/5	−1/14	3/10						
$-2\ +1$	1/14	−3/10	3/7	−1/5						

$0\ -2$	3/14	1/2	2/7								
$-1\ -1$	4/7	0	−3/7								
$-2\ \ 0$	3/14	−1/2	2/7								

$-1\ -2$	1/2	1/2	4									
$-2\ -1$	1/2	−1/2	−4									

$-2\ -2$	1												

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{2}_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{2}_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{2}_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{2}_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{2}_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^{2}_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{2}_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{2}_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{2}_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$d^{0}_{0,0} = \cos\theta$$

$$d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^{1}_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1}_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^{1}_{1,-1} = \frac{1-\cos\theta}{2}$$

Figure 35.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.