# 7. ELECTROMAGNETIC RELATIONS

| Quantity | Gaussian CGS | SI |
|---|---|---|
| Conversion factors:<br>Charge:<br>Potential:<br>Magnetic field: | <br>$2.997\,924\,58 \times 10^9$ esu<br>$(1/299.792\,458)$ statvolt (ergs/esu)<br>$10^4$ gauss $= 10^4$ dyne/esu | <br>$= 1$ C $= 1$ A s<br>$= 1$ V $= 1$ J C$^{-1}$<br>$= 1$ T $= 1$ N A$^{-1}$m$^{-1}$ |
| Lorentz force: | $\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$ | $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ |
| Maxwell equations: | $\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho$<br>$\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$<br>$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$<br>$\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$ | $\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$<br>$\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$<br>$\boldsymbol{\nabla} \cdot \mathbf{B} = 0$<br>$\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$ |
| Constitutive relations: | $\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$ | $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ |
| Linear media:<br>Permitivity of free space:<br>Permeability of free space: | $\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$<br>1<br>1 | $\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$<br>$\epsilon_0 = 8.854\,187\ldots \times 10^{-12}$ F m$^{-1}$<br>$\mu_0 = 4\pi \times 10^{-7}$ N A$^{-2}$ |
| Fields from potentials: | $\mathbf{E} = -\boldsymbol{\nabla}V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$<br>$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ | $\mathbf{E} = -\boldsymbol{\nabla}V - \frac{\partial \mathbf{A}}{\partial t}$<br>$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ |
| Static potentials:<br>(coulomb gauge) | $V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$<br>$\mathbf{A} = \frac{1}{c}\oint \frac{I\, d\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ | $V = \frac{1}{4\pi\epsilon_0}\sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$<br>$\mathbf{A} = \frac{\mu_0}{4\pi}\oint \frac{I\, d\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ |
| Relativistic transformations:<br>($\mathbf{v}$ is the velocity of the primed frame as seen in the unprimed frame) | $\mathbf{E}'_\parallel = \mathbf{E}_\parallel$<br>$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \frac{1}{c}\mathbf{v} \times \mathbf{B})$<br>$\mathbf{B}'_\parallel = \mathbf{B}_\parallel$<br>$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{1}{c}\mathbf{v} \times \mathbf{E})$ | $\mathbf{E}'_\parallel = \mathbf{E}_\parallel$<br>$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})$<br>$\mathbf{B}'_\parallel = \mathbf{B}_\parallel$<br>$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$ |
| $\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7}$ N A$^{-2}$ $= 8.987\,55\ldots \times 10^9$ m F$^{-1}$ ; $\quad \frac{\mu_0}{4\pi} = 10^{-7}$ N A$^{-2}$ ; $\quad c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997\,924\;58 \times 10^8$ m s$^{-1}$ | | |

## 7.1. Impedances (SI units)

$\rho$ = resistivity at room temperature in $10^{-8}\ \Omega$ m:
$\sim 1.7$ for Cu $\sim 5.5$ for W
$\sim 2.4$ for Au $\sim 73$ for SS 304
$\sim 2.8$ for Al $\sim 100$ for Nichrome
(Al alloys may have double the Al value.)

For alternating currents, instantaneous current $I$, voltage $V$, angular frequency $\omega$:

$$V = V_0\ e^{j\omega t} = ZI\ . \tag{7.1}$$

Impedance of self-inductance $L$: $Z = j\omega L$ .

Impedance of capacitance $C$: $Z = 1/j\omega C$ .

Impedance of free space: $Z = \sqrt{\mu_0/\epsilon_0} = 376.7\ \Omega$ .

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\ \rho}{\delta}\ , \quad \text{where } \delta = \text{skin depth}\ ; \tag{7.2}$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6\text{ cm}}{\sqrt{\nu\ (\text{Hz})}} \quad \text{for Cu}\ . \tag{7.3}$$

## 7.2. Capacitance $\widehat{C}$ and inductance $\widehat{L}$ per unit length (SI units) [negligible skin depth]

Flat rectangular plates of width $w$, separated by $d \ll w$ with linear medium $(\epsilon, \mu)$ between:

$$\widehat{C} = \epsilon\ \frac{w}{d}\ ; \qquad \widehat{L} = \mu\ \frac{d}{w}\ ; \tag{7.4}$$

$$\epsilon/\epsilon_0 = 2 \text{ to } 6 \text{ for plastics; } 4 \text{ to } 8 \text{ for porcelain, glasses;} \tag{7.5}$$

$$\mu/\mu_0 \simeq 1\ . \tag{7.6}$$

Coaxial cable of inner radius $r_1$, outer radius $r_2$:

$$\widehat{C} = \frac{2\pi\ \epsilon}{\ln\ (r_2/r_1)}\ ; \qquad \widehat{L} = \frac{\mu}{2\pi}\ \ln\ (r_2/r_1)\ . \tag{7.7}$$

Transmission lines (no loss):

$$\text{Impedance: } Z = \sqrt{\widehat{L}/\widehat{C}}\ . \tag{7.8}$$

$$\text{Velocity: } v = 1/\sqrt{\widehat{L}\ \widehat{C}} = 1/\sqrt{\mu\epsilon}\ . \tag{7.9}$$

## 7.3. Synchrotron radiation (CGS units)

For a particle of charge $e$, velocity $v = \beta c$, and energy $E = \gamma mc^2$, traveling in a circular orbit of radius $R$, the classical energy loss per revolution $\delta E$ is

$$\delta E = \frac{4\pi}{3}\ \frac{e^2}{R}\ \beta^3\ \gamma^4\ . \tag{7.10}$$

For high-energy electrons or positrons ($\beta \approx 1$), this becomes

$$\delta E\ (\text{in MeV}) \approx 0.0885\ \ [E(\text{in GeV})]^4/R(\text{in m})\ . \tag{7.11}$$

For $\gamma \gg 1$, the energy radiated per revolution into the photon energy interval $d(\hbar\omega)$ is

$$dI = \frac{8\pi}{9}\alpha\ \gamma\ F(\omega/\omega_c)\ d(\hbar\omega)\ , \tag{7.12}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \tag{7.13}$$

is the critical frequency. The normalized function $F(y)$ is

$$F(y) = \frac{9}{8\pi}\sqrt{3}\ y\ \int_y^\infty\ K_{5/3}\,(x)\ \ dx\ , \tag{7.14}$$

where $K_{5/3}\,(x)$ is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c\ (\text{in keV}) \approx 2.22\ \ [E(\text{in GeV})]^3/R(\text{in m})\ . \tag{7.15}$$

Fig. 7.1 shows $F(y)$ over the important range of $y$.

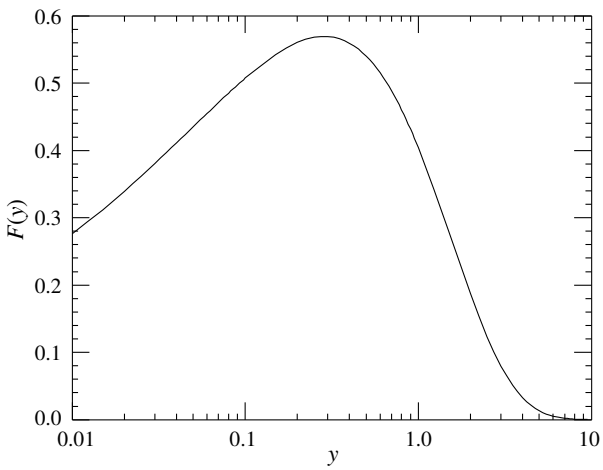


**Figure 7.1:** The normalized synchrotron radiation spectrum $F(y)$.

For $\gamma \gg 1$ and $\omega \ll \omega_c$ ,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha\ (\omega R/c)^{1/3}\ , \tag{7.16}$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c\ ,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}}\ \alpha\ \gamma\ \left(\frac{\omega}{\omega_c}\right)^{1/2}\ e^{-\omega/\omega_c}\ \left[1 + \frac{55}{72}\frac{\omega_c}{\omega} + \ldots\right]\ . \tag{7.17}$$

The radiation is confined to angles $\lesssim 1/\gamma$ relative to the instantaneous direction of motion. For $\gamma \gg 1$, where Eq. (7.12) applies, the mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}}\alpha\gamma\ , \tag{7.18}$$

and the mean energy per photon is

$$\langle\hbar\omega\rangle = \frac{8}{15\sqrt{3}}\hbar\omega_c\ . \tag{7.19}$$

When $\langle\hbar\omega\rangle \gtrsim \mathrm{O}(E)$, quantum corrections are important.

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See J.D. Jackson, *Classical Electrodynamics*, 3rd edition (John Wiley & Sons, New York, 1998) for more formulae and details. (Note that ealier editions had $\omega_c$ twice as large as Eq. (7.13).