# 34. KINEMATICS

Revised January 2000 by J.D. Jackson (LBNL).

Throughout this section units are used in which $\hbar = c = 1$. The following conversions are useful: $\hbar c = 197.3$ MeV fm, $(\hbar c)^2 = 0.3894$ (GeV)$^2$ mb.

## 34.1. Lorentz transformations

The energy $E$ and 3-momentum $\boldsymbol{p}$ of a particle of mass $m$ form a 4-vector $p = (E, \boldsymbol{p})$ whose square $p^2 \equiv E^2 - |\boldsymbol{p}|^2 = m^2$. The velocity of the particle is $\boldsymbol{\beta} = \boldsymbol{p}/E$. The energy and momentum $(E^*, \boldsymbol{p}^*)$ viewed from a frame moving with velocity $\boldsymbol{\beta}_f$ are given by

$$\begin{pmatrix} E^* \\ p_\parallel^* \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f\beta_f \\ -\gamma_f\beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_\parallel \end{pmatrix} , \qquad p_T^* = p_T , \tag{34.1}$$

where $\gamma_f = (1 - \beta_f^2)^{-1/2}$ and $p_T$ $(p_\parallel)$ are the components of $\boldsymbol{p}$ perpendicular (parallel) to $\boldsymbol{\beta}_f$. Other 4-vectors, such as the space-time coordinates of events, of course transform in the same way. The scalar product of two 4-momenta $p_1 \cdot p_2 = E_1E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ is invariant (frame independent).

## 34.2. Center-of-mass energy and momentum

In the collision of two particles of masses $m_1$ and $m_2$ the total center-of-mass energy can be expressed in the Lorentz-invariant form

$$\begin{aligned} E_{\rm cm} &= \left[(E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2\right]^{1/2} , \\ &= \left[m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta)\right]^{1/2} , \end{aligned} \tag{34.2}$$

where $\theta$ is the angle between the particles. In the frame where one particle (of mass $m_2$) is at rest (lab frame),

$$E_{\rm cm} = (m_1^2 + m_2^2 + 2E_{1\,\rm lab}\, m_2)^{1/2} . \tag{34.3}$$

The velocity of the center-of-mass in the lab frame is

$$\boldsymbol{\beta}_{\rm cm} = \boldsymbol{p}_{\rm lab}/(E_{1\,\rm lab} + m_2) , \tag{34.4}$$

where $\boldsymbol{p}_{\rm lab} \equiv \boldsymbol{p}_{1\,\rm lab}$ and

$$\gamma_{\rm cm} = (E_{1\,\rm lab} + m_2)/E_{\rm cm} . \tag{34.5}$$

The c.m. momenta of particles 1 and 2 are of magnitude

$$p_{\rm cm} = p_{\rm lab}\frac{m_2}{E_{\rm cm}} . \tag{34.6}$$

For example, if a 0.80 GeV/$c$ kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/$c$. It is also useful to note that

$$E_{\rm cm}\, dE_{\rm cm} = m_2\, dE_{1\,\rm lab} = m_2\, \beta_{1\,\rm lab}\, dp_{\rm lab} . \tag{34.7}$$

## 34.3. Lorentz-invariant amplitudes

The matrix elements for a scattering or decay process are written in terms of an invariant amplitude $-i\mathscr{M}$. As an example, the $S$-matrix for $2 \to 2$ scattering is related to $\mathscr{M}$ by

$$\begin{aligned} \langle p_1'p_2'\,|S|\,p_1p_2\rangle &= I - i(2\pi)^4\,\delta^4(p_1 + p_2 - p_1' - p_2') \\ &\times \frac{\mathscr{M}(p_1, p_2;\, p_1', p_2')}{(2E_1)^{1/2}\,(2E_2)^{1/2}\,(2E_1')^{1/2}\,(2E_2')^{1/2}} . \end{aligned} \tag{34.8}$$

The state normalization is such that

$$\langle p'|p\rangle = (2\pi)^3\delta^3(\boldsymbol{p} - \boldsymbol{p}') . \tag{34.9}$$

## 34.4. Particle decays

The partial decay rate of a particle of mass $M$ into $n$ bodies in its rest frame is given in terms of the Lorentz-invariant matrix element $\mathscr{M}$ by

$$d\Gamma = \frac{(2\pi)^4}{2M}\,|\mathscr{M}|^2\, d\Phi_n\,(P;\, p_1, \ldots, p_n), \tag{34.10}$$

where $d\Phi_n$ is an element of $n$-body phase space given by

$$d\Phi_n(P;\, p_1, \ldots, p_n) = \delta^4\,(P - \sum_{i=1}^n p_i)\,\prod_{i=1}^n \frac{d^3p_i}{(2\pi)^3 2E_i} . \tag{34.11}$$

This phase space can be generated recursively, viz.

$$\begin{aligned} d\Phi_n(P;\, p_1, \ldots, p_n) &= d\Phi_j(q;\, p_1, \ldots, p_j) \\ &\times\ d\Phi_{n-j+1}\,(P;\, q, p_{j+1}, \ldots, p_n)(2\pi)^3 dq^2 , \end{aligned} \tag{34.12}$$

where $q^2 = (\sum_{i=1}^j E_i)^2 - \left|\sum_{i=1}^j \boldsymbol{p}_i\right|^2$. This form is particularly useful in the case where a particle decays into another particle that subsequently decays.

**34.4.1.** ***Survival probability***: If a particle of mass $M$ has mean proper lifetime $\tau$ $(= 1/\Gamma)$ and has momentum $(E, \boldsymbol{p})$, then the probability that it lives for a time $t_0$ or greater before decaying is given by

$$P(t_0) = e^{-t_0\,\Gamma/\gamma} = e^{-Mt_0\,\Gamma/E} , \tag{34.13}$$

and the probability that it travels a distance $x_0$ or greater is

$$P(x_0) = e^{-Mx_0\,\Gamma/|\boldsymbol{p}|} . \tag{34.14}$$

**34.4.2.** ***Two-body decays***:

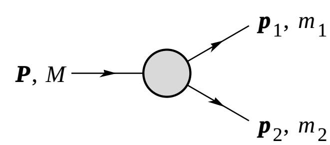


**Figure 34.1:** Definitions of variables for two-body decays.

In the rest frame of a particle of mass $M$, decaying into 2 particles labeled 1 and 2,

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M} , \tag{34.15}$$

$$\begin{aligned} |\boldsymbol{p}_1| &= |\boldsymbol{p}_2| \\ &= \frac{\left[\left(M^2 - (m_1 + m_2)^2\right)\left(M^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2M} , \end{aligned} \tag{34.16}$$

and

$$d\Gamma = \frac{1}{32\pi^2}\,|\mathscr{M}|^2\,\frac{|\boldsymbol{p}_1|}{M^2}\,d\Omega , \tag{34.17}$$

where $d\Omega = d\phi_1 d(\cos\theta_1)$ is the solid angle of particle 1.

**34.4.3.** ***Three-body decays***:

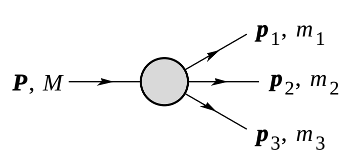


**Figure 34.2:** Definitions of variables for three-body decays.

Defining $p_{ij} = p_i + p_j$ and $m_{ij}^2 = p_{ij}^2$, then $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$ and $m_{12}^2 = (P - p_3)^2 = M^2 + m_3^2 - 2ME_3$, where $E_3$ is the energy of particle 3 in the rest frame of $M$. In that frame, the momenta of the three decay particles lie in a plane. The relative orientation of these three momenta is fixed if their energies are known. The momenta can therefore be specified in space by giving three Euler angles $(\alpha, \beta, \gamma)$ that specify the orientation of the final system relative to the initial particle [1]. Then

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} |\mathscr{M}|^2 \, dE_1 \, dE_2 \, d\alpha \, d(\cos\beta) \, d\gamma \, . \tag{34.18}$$

Alternatively

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} |\mathscr{M}|^2 \, |\boldsymbol{p}_1^*| \, |\boldsymbol{p}_3| \, dm_{12} \, d\Omega_1^* \, d\Omega_3 \, , \tag{34.19}$$

where $(|\boldsymbol{p}_1^*|, \Omega_1^*)$ is the momentum of particle 1 in the rest frame of 1 and 2, and $\Omega_3$ is the angle of particle 3 in the rest frame of the decaying particle. $|\boldsymbol{p}_1^*|$ and $|\boldsymbol{p}_3|$ are given by

$$|\boldsymbol{p}_1^*| = \frac{\left[\left(m_{12}^2 - (m_1 + m_2)^2\right)\left(m_{12}^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2m_{12}} \, , \tag{34.20$a$}$$

and

$$|\boldsymbol{p}_3| = \frac{\left[\left(M^2 - (m_{12} + m_3)^2\right)\left(M^2 - (m_{12} - m_3)^2\right)\right]^{1/2}}{2M} \, . \tag{34.20$b$}$$

[Compare with Eq. (34.16).]

If the decaying particle is a scalar or we average over its spin states, then integration over the angles in Eq. (34.18) gives

$$d\Gamma = \frac{1}{(2\pi)^3} \frac{1}{8M} \overline{|\mathscr{M}|^2} \, dE_1 \, dE_2$$

$$= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \overline{|\mathscr{M}|^2} \, dm_{12}^2 \, dm_{23}^2 \, . \tag{34.21}$$

This is the standard form for the Dalitz plot.

**34.4.3.1.** *Dalitz plot:* For a given value of $m_{12}^2$, the range of $m_{23}^2$ is determined by its values when $\boldsymbol{p}_2$ is parallel or antiparallel to $\boldsymbol{p}_3$:

$$(m_{23}^2)_{\max} =$$

$$(E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2 \, , \tag{34.22$a$}$$

$$(m_{23}^2)_{\min} =$$

$$(E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2 \, . \tag{34.22$b$}$$

Here $E_2^* = (m_{12}^2 - m_1^2 + m_2^2)/2m_{12}$ and $E_3^* = (M^2 - m_{12}^2 - m_3^2)/2m_{12}$ are the energies of particles 2 and 3 in the $m_{12}$ rest frame. The scatter plot in $m_{12}^2$ and $m_{23}^2$ is called a Dalitz plot. If $\overline{|\mathscr{M}|^2}$ is constant, the allowed region of the plot will be uniformly populated with events [see Eq. (34.21)]. A nonuniformity in the plot gives immediate information on $|\mathscr{M}|^2$. For example, in the case of $D \to K\pi\pi$, bands appear when $m_{(K\pi)} = m_{K^*(892)}$, reflecting the appearance of the decay chain $D \to K^*(892)\pi \to K\pi\pi$.

**34.4.4.** ***Kinematic limits***: In a three-body decay the maximum of $|\boldsymbol{p}_3|$, [given by Eq. (34.20)], is achieved when $m_{12} = m_1 + m_2$, *i.e.*, particles 1 and 2 have the same vector velocity in the rest frame of the decaying particle. If, in addition, $m_3 > m_1, m_2$, then $|\boldsymbol{p}_3|_{\max} > |\boldsymbol{p}_1|_{\max}$, $|\boldsymbol{p}_2|_{\max}$.

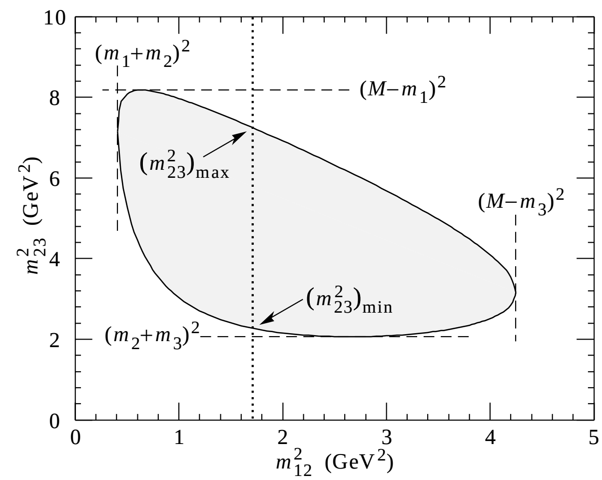


**Figure 34.3:** Dalitz plot for a three-body final state. In this example, the state is $\pi^+ \overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

**34.4.5.** ***Multibody decays***: The above results may be generalized to final states containing any number of particles by combining some of the particles into "effective particles" and treating the final states as 2 or 3 "effective particle" states. Thus, if $p_{ijk\ldots} = p_i + p_j + p_k + \ldots$, then

$$m_{ijk\ldots} = \sqrt{p^2{}_{ijk\ldots}} \, , \tag{34.23}$$

and $m_{ijk\ldots}$ may be used in place of *e.g.*, $m_{12}$ in the relations in Sec. 34.4.3 or 34.4.3.1 above.

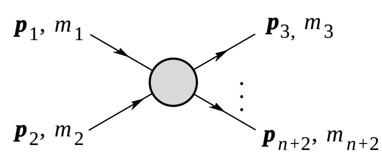


**Figure 34.4:** Definitions of variables for production of an $n$-body final state.

## 34.5. Cross sections

The differential cross section is given by

$$d\sigma = \frac{(2\pi)^4 |\mathscr{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}$$

$$\times \; d\Phi_n(p_1 + p_2; \, p_3, \ldots, p_{n+2}) \, . \tag{34.24}$$

[See Eq. (34.11).] In the rest frame of $m_2$(lab),

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = m_2 p_{1\,\mathrm{lab}} \, ; \tag{34.25$a$}$$

while in the center-of-mass frame

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = p_{1\mathrm{cm}} \sqrt{s} \, . \tag{34.25$b$}$$

**34.5.1.** ***Two-body reactions***:

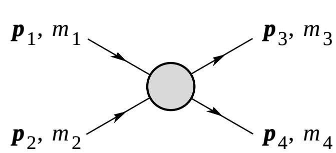


**Figure 34.5:** Definitions of variables for a two-body final state.

Two particles of momenta $p_1$ and $p_2$ and masses $m_1$ and $m_2$ scatter to particles of momenta $p_3$ and $p_4$ and masses $m_3$ and $m_4$; the Lorentz-invariant Mandelstam variables are defined by

$$\begin{aligned} s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\ &= m_1^2 + 2E_1E_2 - 2\boldsymbol{p}_1\cdot\boldsymbol{p}_2 + m_2^2 \ , \qquad (34.26) \\ t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\ &= m_1^2 - 2E_1E_3 + 2\boldsymbol{p}_1\cdot\boldsymbol{p}_3 + m_3^2 \ , \qquad (34.27) \\ u &= (p_1-p_4)^2 = (p_2-p_3)^2 \\ &= m_1^2 - 2E_1E_4 + 2\boldsymbol{p}_1\cdot\boldsymbol{p}_4 + m_4^2 \ , \qquad (34.28) \end{aligned}$$

and they satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \qquad (34.29)$$

The two-body cross section may be written as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s}\,\frac{1}{|\boldsymbol{p}_{1\text{cm}}|^2}\,|\mathscr{M}|^2 \ . \qquad (34.30)$$

In the center-of-mass frame

$$\begin{aligned} t &= (E_{1\text{cm}} - E_{3\text{cm}})^2 - (p_{1\text{cm}} - p_{3\text{cm}})^2 \qquad - 4p_{1\text{cm}}\,p_{3\text{cm}}\,\sin^2(\theta_{\text{cm}}/2) \\ &= t_0 - 4p_{1\text{cm}}\,p_{3\text{cm}}\,\sin^2(\theta_{\text{cm}}/2) \ , \qquad (34.31) \end{aligned}$$

where $\theta_{\text{cm}}$ is the angle between particle 1 and 3. The limiting values $t_0$ $(\theta_{\text{cm}}=0)$ and $t_1$ $(\theta_{\text{cm}}=\pi)$ for $2\to 2$ scattering are

$$t_0(t_1) = \left[\frac{m_1^2 - m_3^2 - m_2^2 + m_4^2}{2\sqrt{s}}\right]^2 - (p_{1\,\text{cm}} \mp p_{3\,\text{cm}})^2 \ . \qquad (34.32)$$

In the literature the notation $t_{\text{min}}$ $(t_{\text{max}})$ for $t_0$ $(t_1)$ is sometimes used, which should be discouraged since $t_0 > t_1$. The center-of-mass energies and momenta of the incoming particles are

$$E_{1\text{cm}} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \ , \qquad E_{2\text{cm}} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \ , \qquad (34.33)$$

For $E_{3\text{cm}}$ and $E_{4\text{cm}}$, change $m_1$ to $m_3$ and $m_2$ to $m_4$. Then

$$p_{i\,\text{cm}} = \sqrt{E_{i\,\text{cm}}^2 - m_i^2} \text{ and } p_{1\text{cm}} = \frac{p_{1\,\text{lab}}\,m_2}{\sqrt{s}} \ . \qquad (34.34)$$

Here the subscript lab refers to the frame where particle 2 is at rest. [For other relations see Eqs. (34.2)–(34.4).]

**34.5.2.** ***Inclusive reactions***: Choose some direction (usually the beam direction) for the $z$-axis; then the energy and momentum of a particle can be written as

$$E = m_T\cosh y \ , \ p_x \ , \ p_y \ , \ p_z = m_T\sinh y \ , \qquad (34.35)$$

where $m_T$ is the transverse mass

$$m_T^2 = m^2 + p_x^2 + p_y^2 \ , \qquad (34.36)$$

and the rapidity $y$ is defined by

$$\begin{aligned} y &= \frac{1}{2}\ln\left(\frac{E+p_z}{E-p_z}\right) \\ &= \ln\left(\frac{E+p_z}{m_T}\right) = \tanh^{-1}\left(\frac{p_z}{E}\right) \ . \qquad (34.37) \end{aligned}$$

Under a boost in the $z$-direction to a frame with velocity $\beta$, $y \to y - \tanh^{-1}\beta$. Hence the shape of the rapidity distribution $dN/dy$ is invariant. The invariant cross section may also be rewritten

$$E\frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{d\phi\,dy\,p_T\,dp_T} \Longrightarrow \frac{d^2\sigma}{\pi\,dy\,d(p_T^2)} \ . \qquad (34.38)$$

The second form is obtained using the identity $dy/dp_z = 1/E$, and the third form represents the average over $\phi$.

Feynman's $x$ variable is given by

$$x = \frac{p_z}{p_{z\,\text{max}}} \approx \frac{E+p_z}{(E+p_z)_{\text{max}}} \quad (p_T \ll |p_z|) \ . \qquad (34.39)$$

In the c.m. frame,

$$x \approx \frac{2p_{z\,\text{cm}}}{\sqrt{s}} = \frac{2m_T\sinh y_{\text{cm}}}{\sqrt{s}} \qquad (34.40)$$

and

$$= (y_{\text{cm}})_{\text{max}} = \ln(\sqrt{s}/m) \ . \qquad (34.41)$$

For $p \gg m$, the rapidity [Eq. (34.37)] may be expanded to obtain

$$\begin{aligned} y &= \frac{1}{2}\ln\frac{\cos^2(\theta/2) + m^2/4p^2 + \ldots}{\sin^2(\theta/2) + m^2/4p^2 + \ldots} \\ &\approx -\ln\tan(\theta/2) \equiv \eta \qquad (34.42) \end{aligned}$$

where $\cos\theta = p_z/p$. The pseudorapidity $\eta$ defined by the second line is approximately equal to the rapidity $y$ for $p \gg m$ and $\theta \gg 1/\gamma$, and in any case can be measured when the mass and momentum of the particle is unknown. From the definition one can obtain the identities

$$\sinh\eta = \cot\theta \ , \quad \cosh\eta = 1/\sin\theta \ , \quad \tanh\eta = \cos\theta \ . \qquad (34.43)$$

**34.5.3.** ***Partial waves***: The amplitude in the center of mass for elastic scattering of spinless particles may be expanded in Legendre polynomials

$$f(k,\theta) = \frac{1}{k}\sum_\ell (2\ell+1)a_\ell P_\ell(\cos\theta) \ , \qquad (34.44)$$

where $k$ is the c.m. momentum, $\theta$ is the c.m. scattering angle, $a_\ell = (\eta_\ell e^{2i\delta_\ell} - 1)/2i$, $0 \le \eta_\ell \le 1$, and $\delta_\ell$ is the phase shift of the $\ell^{th}$ partial wave. For purely elastic scattering, $\eta_\ell = 1$. The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \ . \qquad (34.45)$$

The optical theorem states that

$$\sigma_{\text{tot}} = \frac{4\pi}{k}\text{Im}\,f(k,0) \ , \qquad (34.46)$$

and the cross section in the $\ell^{th}$ partial wave is therefore bounded:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)|a_\ell|^2 \le \frac{4\pi(2\ell+1)}{k^2} \ . \qquad (34.47)$$

The evolution with energy of a partial-wave amplitude $a_\ell$ can be displayed as a trajectory in an Argand plot, as shown in Fig. 34.6.

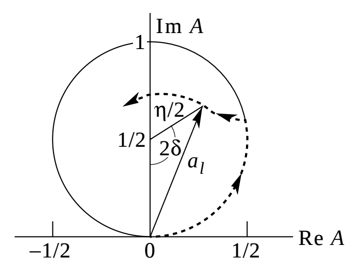


**Figure 34.6:** Argand plot showing a partial-wave amplitude $a_\ell$ as a function of energy. The amplitude leaves the unitary circle where inelasticity sets in ($\eta_\ell < 1$).

The usual Lorentz-invariant matrix element $\mathscr{M}$ (see Sec. 34.3 above) for the elastic process is related to $f(k,\theta)$ by

$$\mathscr{M} = -8\pi\sqrt{s}\,f(k,\theta) \ , \qquad (34.48)$$

so

$$\sigma_{\text{tot}} = -\frac{1}{2p_{\text{lab}}\,m_2}\,\text{Im}\,\mathscr{M}(t=0) \ , \qquad (34.49)$$

where $s$ and $t$ are the center-of-mass energy squared and momentum transfer squared, respectively (see Sec. 34.4.1).

**34.5.3.1.** *Resonances:* The Breit-Wigner (nonrelativistic) form for an elastic amplitude $a_\ell$ with a resonance at c.m. energy $E_R$, elastic width $\Gamma_{\rm el}$, and total width $\Gamma_{\rm tot}$ is

$$a_\ell = \frac{\Gamma_{\rm el}/2}{E_R - E - i\Gamma_{\rm tot}/2} \ , \qquad (34.50)$$

where $E$ is the c.m. energy. As shown in Fig. 34.7, in the absence of background the elastic amplitude traces a counterclockwise circle with center $ix_{\rm el}/2$ and radius $x_{\rm el}/2$, where the elasticity $x_{\rm el} = \Gamma_{\rm el}/\Gamma_{\rm tot}$. The amplitude has a pole at $E = E_R - i\Gamma_{\rm tot}/2$.

The spin-averaged Breit-Wigner cross section for a spin-$J$ resonance produced in the collision of particles of spin $S_1$ and $S_2$ is

$$\sigma_{BW}(E) = \frac{(2J+1)}{(2S_1+1)(2S_2+1)} \frac{\pi}{k^2} \frac{B_{\rm in} B_{\rm out} \Gamma_{\rm tot}^2}{(E-E_R)^2 + \Gamma_{\rm tot}^2/4} \ , \qquad (34.51)$$

where $k$ is the c.m. momentum, $E$ is the c.m. energy, and $B_{\rm in}$ and $B_{\rm out}$ are the branching fractions of the resonance into the entrance and exit channels. The $2S+1$ factors are the multiplicities of the incident spin states, and are replaced by 2 for photons. This expression is valid only for an isolated state. If the width is not small, $\Gamma_{\rm tot}$ cannot be treated as a constant independent of $E$. There are many other forms for $\sigma_{BW}$, all of which are equivalent to the one given here in the narrow-width case. Some of these forms may be more appropriate if the resonance is broad.

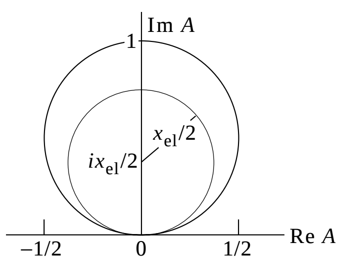


**Figure 34.7:** Argand plot for a resonance.

The relativistic Breit-Wigner form corresponding to Eq. (34.50) is:

$$a_\ell = \frac{-m\Gamma_{\rm el}}{s - m^2 + im\Gamma_{\rm tot}} \ . \qquad (34.52)$$

A better form incorporates the known kinematic dependences, replacing $m\Gamma_{\rm tot}$ by $\sqrt{s}\,\Gamma_{\rm tot}(s)$, where $\Gamma_{\rm tot}(s)$ is the width the resonance particle would have if its mass were $\sqrt{s}$, and correspondingly $m\Gamma_{\rm el}$ by $\sqrt{s}\,\Gamma_{\rm el}(s)$ where $\Gamma_{\rm el}(s)$ is the partial width in the incident channel for a mass $\sqrt{s}$:

$$a_\ell = \frac{-\sqrt{s}\,\Gamma_{\rm el}(s)}{s - m^2 + i\sqrt{s}\,\Gamma_{\rm tot}(s)} \ . \qquad (34.53)$$

For the $Z$ boson, all the decays are to particles whose masses are small enough to be ignored, so on dimensional grounds $\Gamma_{\rm tot}(s) = \sqrt{s}\,\Gamma_0/m_Z$, where $\Gamma_0$ defines the width of the $Z$, and $\Gamma_{\rm el}(s)/\Gamma_{\rm tot}(s)$ is constant. A full treatment of the line shape requires consideration of dynamics, not just kinematics. For the $Z$ this is done by calculating the radiative corrections in the Standard Model.

**References:**

1. See, for example, J.J. Sakurai, *Modern Quantum Mechnaics*, Addison-Wesley (1985), p. 172, or D.M. Brink and G.R. Satchler, *Angular Momentum*, 2nd ed., Oxford University Press (1968), p. 20.