7. ELECTROMAGNETIC RELATIONS

Quantity	Gaussian CGS	SI
Conversion factors:		
Charge:	$2.997\,924\,58 \times 10^9$ esu	$= 1$ C $= 1$ A s
Potential:	$(1/299.792\,458)$ statvolt (ergs/esu)	$= 1$ V $= 1$ J C^{-1}
Magnetic field:	10^4 gauss $= 10^4$ dyne/esu	$= 1$ T $= 1$ N $\mathrm{A}^{-1}\mathrm{m}^{-1}$
Lorentz force:	$\mathbf{F} = q\,(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$	$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$
Maxwell equations:	$\boldsymbol{\nabla} \cdot \mathbf{D} = 4\pi\rho$ $\boldsymbol{\nabla} \times \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi}{c}\mathbf{J}$ $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ $\boldsymbol{\nabla} \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0$	$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho$ $\boldsymbol{\nabla} \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J}$ $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ $\boldsymbol{\nabla} \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$
Constitutive relations:	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}, \quad \mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$	$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}, \quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$
Linear media:	$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$	$\mathbf{D} = \epsilon\mathbf{E}, \quad \mathbf{H} = \mathbf{B}/\mu$
Permitivity of free space:	1	$\epsilon_0 = 8.854\,187\ldots \times 10^{-12}$ F m^{-1}
Permeability of free space:	1	$\mu_0 = 4\pi \times 10^{-7}$ N A^{-2}
Fields from potentials:	$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$	$\mathbf{E} = -\boldsymbol{\nabla} V - \frac{\partial \mathbf{A}}{\partial t}$ $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$
Static potentials: (coulomb gauge)	$V = \sum_{\text{charges}} \frac{q_i}{r_i} = \int \frac{\rho\,(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ $\mathbf{A} = \frac{1}{c}\oint \frac{I\,\mathbf{d}\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$	$V = \frac{1}{4\pi\epsilon_0} \sum_{\text{charges}} \frac{q_i}{r_i} = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\,(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$ $\mathbf{A} = \frac{\mu_0}{4\pi}\oint \frac{I\,\mathbf{d}\boldsymbol{\ell}}{\lvert\mathbf{r} - \mathbf{r}'\rvert} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}')}{\lvert\mathbf{r} - \mathbf{r}'\rvert}\, d^3x'$
Relativistic transformations: ($\mathbf{v}$ is the velocity of the primed frame as seen in the unprimed frame)	$\mathbf{E}'_\parallel = \mathbf{E}_\parallel$ $\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \frac{1}{c}\mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_\parallel = \mathbf{B}_\parallel$ $\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{1}{c}\mathbf{v} \times \mathbf{E})$	$\mathbf{E}'_\parallel = \mathbf{E}_\parallel$ $\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B})$ $\mathbf{B}'_\parallel = \mathbf{B}_\parallel$ $\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v} \times \mathbf{E})$

$$\frac{1}{4\pi\epsilon_0} = c^2 \times 10^{-7}\ \mathrm{N\,A^{-2}} = 8.987\,55\ldots \times 10^9\ \mathrm{m\,F^{-1}}\ ; \qquad \frac{\mu_0}{4\pi} = 10^{-7}\ \mathrm{N\,A^{-2}}\ ; \qquad c = \frac{1}{\sqrt{\mu_0\epsilon_0}} = 2.997\,924\ 58 \times 10^8\ \mathrm{m\,s^{-1}}$$

7.1. Impedances (SI units)

ρ = resistivity at room temperature in $10^{-8}\,\Omega$ m:
~ 1.7 for Cu $\quad\sim 5.5$ for W
~ 2.4 for Au $\quad\sim 73$ for SS 304
~ 2.8 for Al $\quad\sim 100$ for Nichrome
(Al alloys may have double the Al value.)

For alternating currents, instantaneous current I, voltage V, angular frequency ω:

$$V = V_0\, e^{j\omega t} = ZI\ . \tag{7.1}$$

Impedance of self-inductance L: $Z = j\omega L$.

Impedance of capacitance C: $Z = 1/j\omega C$.

Impedance of free space: $Z = \sqrt{\mu_0/\epsilon_0} = 376.7\ \Omega$.

High-frequency surface impedance of a good conductor:

$$Z = \frac{(1+j)\,\rho}{\delta}\ , \quad \text{where } \delta = \text{skin depth}\ ; \tag{7.2}$$

$$\delta = \sqrt{\frac{\rho}{\pi\nu\mu}} \approx \frac{6.6\text{ cm}}{\sqrt{\nu\,(\text{Hz})}} \quad \text{for Cu}\ . \tag{7.3}$$

7.2. Capacitance $\widehat{C}$ and inductance $\widehat{L}$ per unit length (SI units) [negligible skin depth]

Flat rectangular plates of width w, separated by $d \ll w$ with linear medium (ϵ, μ) between:

$$\widehat{C} = \epsilon\,\frac{w}{d}\ ; \qquad \widehat{L} = \mu\,\frac{d}{w}\ ; \tag{7.4}$$

$$\epsilon/\epsilon_0 = 2 \text{ to } 6 \text{ for plastics; } 4 \text{ to } 8 \text{ for porcelain, glasses;} \tag{7.5}$$

$$\mu/\mu_0 \simeq 1\ . \tag{7.6}$$

Coaxial cable of inner radius r_1, outer radius r_2:

$$\widehat{C} = \frac{2\pi\,\epsilon}{\ln\,(r_2/r_1)}\ ; \quad \widehat{L} = \frac{\mu}{2\pi}\ \ln\,(r_2/r_1)\ . \tag{7.7}$$

Transmission lines (no loss):

$$\text{Impedance: } Z = \sqrt{\widehat{L}/\widehat{C}}\ . \tag{7.8}$$

$$\text{Velocity: } v = 1/\sqrt{\widehat{L}\,\widehat{C}} = 1/\sqrt{\mu\,\epsilon}\ . \tag{7.9}$$

7.3. Synchrotron radiation (CGS units)

For a particle of charge e, velocity $v = \beta c$, and energy $E = \gamma mc^2$, traveling in a circular orbit of radius R, the classical energy loss per revolution δE is

$$\delta E = \frac{4\pi}{3}\,\frac{e^2}{R}\,\beta^3\,\gamma^4\ . \tag{7.10}$$

For high-energy electrons or positrons ($\beta \approx 1$), this becomes

$$\delta E\ (\text{in MeV}) \approx 0.0885\ [E(\text{in GeV})]^4/R(\text{in m})\ . \tag{7.11}$$

For $\gamma \gg 1$, the energy radiated per revolution into the photon energy interval $d(\hbar\omega)$ is

$$dI = \frac{8\pi}{9}\alpha\,\gamma\ F(\omega/\omega_c)\ d(\hbar\omega)\ , \tag{7.12}$$

where $\alpha = e^2/\hbar c$ is the fine-structure constant and

$$\omega_c = \frac{3\gamma^3 c}{2R} \tag{7.13}$$

is the critical frequency. The normalized function $F(y)$ is

$$F(y) = \frac{9}{8\pi}\sqrt{3}\ y \int_y^\infty K_{5/3}\,(x)\ \ dx\ , \tag{7.14}$$

where $K_{5/3}\,(x)$ is a modified Bessel function of the third kind. For electrons or positrons,

$$\hbar\omega_c\,(\text{in keV}) \approx 2.22\ \ [E(\text{in GeV})]^3/R(\text{in m})\ . \tag{7.15}$$

Fig. 7.1 shows $F(y)$ over the important range of y.

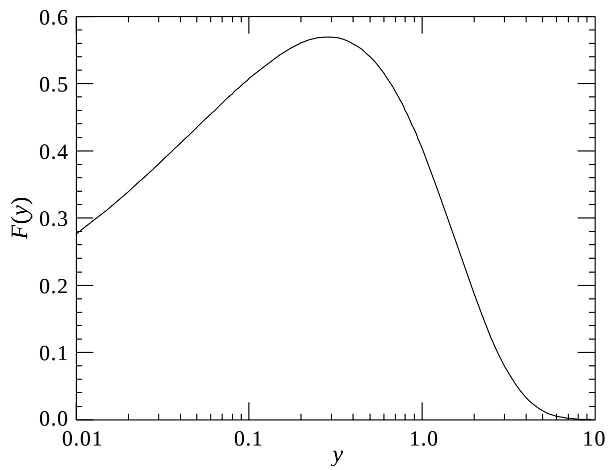

Figure 7.1: The normalized synchrotron radiation spectrum $F(y)$.

For $\gamma \gg 1$ and $\omega \ll \omega_c$,

$$\frac{dI}{d(\hbar\omega)} \approx 3.3\alpha\,(\omega R/c)^{1/3}\ , \tag{7.16}$$

whereas for

$$\gamma \gg 1 \text{ and } \omega \gtrsim 3\omega_c\ ,$$

$$\frac{dI}{d(\hbar\omega)} \approx \sqrt{\frac{3\pi}{2}}\ \alpha\,\gamma\left(\frac{\omega}{\omega_c}\right)^{1/2} e^{-\omega/\omega_c}\left[1 + \frac{55}{72}\frac{\omega_c}{\omega} + \ldots\right]\ . \tag{7.17}$$

The radiation is confined to angles $\lesssim 1/\gamma$ relative to the instantaneous direction of motion. The mean number of photons emitted per revolution is

$$N_\gamma = \frac{5\pi}{\sqrt{3}}\alpha\gamma\ , \tag{7.18}$$

and the mean energy per photon is

$$\langle\hbar\omega\rangle = \frac{8}{15\sqrt{3}}\hbar\omega_c\ . \tag{7.19}$$

When $\langle\hbar\omega\rangle \gtrsim \mathrm{O}(E)$, quantum corrections are important.

See J.D. Jackson, *Classical Electrodynamics*, 2nd edition (John Wiley & Sons, New York, 1975) for more formulae and details. In his book, Jackson uses a definition of ω_c that is twice as large as the customary one given above.