## MUON DECAY PARAMETERS

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***Introduction:*** All measurements in direct muon decay, $\mu^- \to e^- + 2$ neutrals, and its inverse, $\nu_\mu + e^- \to \mu^- +$ neutral, are successfully described by the "$V$-$A$ interaction", which is a particular case of a local, derivative-free, lepton-number-conserving, four fermion interaction [1]. As shown below, within this framework, the Standard Model assumptions, such as the $V$-$A$ form and the nature of the neutrals ($\nu_\mu$ and $\bar{\nu}_e$), and hence the doublet assignments $(\nu_e\ e^-)_L$ and $(\nu_\mu\ \mu^-)_L$, have been determined from experiments [2,3]. All considerations on muon decay are valid for the leptonic tau decays $\tau \to \ell + \nu_\tau + \bar{\nu}_e$ with the replacements $m_\mu \to m_\tau$, $m_e \to m_\ell$.

***Parameters:*** The differential decay probability to obtain an $e^\pm$ with (reduced) energy between $x$ and $x + dx$, emitted in the direction $\widehat{z}$ at an angle between $\vartheta$ and $\vartheta + d\vartheta$ with respect to the muon polarization vector $\vec{P}_\mu$, and with its spin pointing in the arbitrary direction $\widehat{\zeta}$, neglecting radiative corrections, is given by

$$\begin{aligned}\frac{d^2\Gamma}{dx\ d\cos\vartheta} =& \frac{m_\mu}{4\pi^3}\ W_{e\mu}^4\ G_F^2\ \sqrt{x^2 - x_0^2} \\ & \times (F_{IS}(x) \pm P_\mu \cos\vartheta\ F_{AS}(x)) \\ & \times \left[1 + \vec{P}_e(x,\vartheta)\cdot\widehat{\zeta}\right]\ . \end{aligned} \tag{1}$$

Here, $W_{e\mu} = \max(E_e) = (m_\mu^2 + m_e^2)/2m_\mu$ is the maximum $e^\pm$ energy, $x = E_e/W_{e\mu}$ is the reduced energy, $x_0 = m_e/W_{e\mu} = 9.67 \times 10^{-3}$, and $P_\mu = |\vec{P}_\mu|$ is the degree of muon polarization. $\widehat{\zeta}$ is the direction in which a perfect polarization-sensitive electron detector is most sensitive. The isotropic part of the spectrum, $F_{IS}(x)$, the anisotropic part $F_{AS}(x)$ and the electron polarization, $\vec{P}_e(x,\vartheta)$, may be parametrized by the Michel parameters [1,4] $\rho$, $\eta$, $\xi$, $\delta$, *etc.* These are bilinear combinations of the coupling constants $g_{\varepsilon\mu}^\gamma$, which occur in the matrix element (given below).

If the masses of the neutrinos as well as $x_0^2$ are neglected, the energy and angular distribution of the electron in the rest

frame of a muon ($\mu^\pm$) measured by a polarization insensitive detector, is given by

$$\frac{d^2\Gamma}{dx\ d\cos\vartheta} \sim x^2 \cdot \Big\{ 3(1-x) + \frac{2\rho}{3}(4x-3) + 3\eta\ x_0(1-x)/x \\ \pm P_\mu \cdot \xi \cdot \cos\vartheta \left[ 1 - x + \frac{2\delta}{3}(4x-3) \right] \Big\} \ .$$

Here, $\vartheta$ is the angle between the electron momentum and the muon spin, and $x \equiv 2E_e/m_\mu$. For the Standard Model coupling, we obtain $\rho = \xi\delta = 3/4$, $\xi = 1$, $\eta = 0$ and the differential decay rate is

$$\frac{d^2\Gamma}{dx\ \ d\cos\vartheta} \ = \ \frac{G_F^2 m_\mu^5}{192\pi^3}\ [3 - 2x \pm P_\mu \cos\vartheta(2x-1)]\ x^2 \quad .$$

The coefficient in front of the square bracket is the total decay rate.

If only the neutrino masses are neglected, and if the $e^\pm$ polarization is detected, then the functions in Eq. (1) become

$$F_{IS}(x) = x(1-x) + \tfrac{2}{9}\ \rho(4x^2 - 3x - x_0^2) + \eta \cdot x_0(1-x)$$
$$F_{AS}(x) = \tfrac{1}{3}\xi\ \sqrt{x^2 - x_0^2} \\ \times \left[ 1 - x + \tfrac{2}{3}\delta\Big(4x - 3 + \Big(\sqrt{1-x_0^2} - 1\Big)\Big) \right]$$
$$\vec{P}_e(x,\vartheta) = P_{T_1}\ \widehat{x} + P_{T_2}\widehat{y} + P_L\ \widehat{z}\ .$$

Here $\widehat{x}$, $\widehat{y}$, and $\widehat{z}$ are orthogonal unit vectors defined as follows:

| | |
|---|---|
| $\widehat{z}$ | is along the $e$ momentum |
| $\widehat{y} = [\widehat{z} \times \vec{P}_\mu]/\lvert[\widehat{z} \times \vec{P}_\mu]\rvert$ | is transverse to the $e$ momentum and perpendicular to the "decay plane" |
| $\widehat{x} = \widehat{y} \times \widehat{z}$ | is transverse to the $e$ momentum and in the "decay plane." |

The components of $\vec{P}_e$ then are given by

$$
\begin{aligned}
P_{T_1}(x,\vartheta) &= P_\mu \sin\vartheta \; F_{T_1}(x)/\left(F_{IS}(x) \pm P_\mu \cos\vartheta \; F_{AS}(x)\right) \\
P_{T_2}(x,\vartheta) &= P_\mu \sin\vartheta \; F_{T_2}(x)/\left(F_{IS}(x) \pm P_\mu \; cos\vartheta \; F_{AS}(x)\right) \\
P_L(x,\vartheta) &= \pm F_{IP}(x) + P_\mu \cos\vartheta \\
&\qquad \times F_{AP}(x)/\left(F_{IS}(x) \pm P_\mu \cos\vartheta \; F_{AS}(x)\right) \; ,
\end{aligned}
$$

where

$$
\begin{aligned}
F_{T_1}(x) &= \tfrac{1}{12}\Big\{-2\left[\xi'' + 12(\rho - \tfrac{3}{4})\right](1-x)x_0 \\
&\qquad -3\eta(x^2 - x_0^2) + \eta''(-3x^2 + 4x - x_0^2)\Big\} \\
F_{T_2}(x) &= \tfrac{1}{3}\sqrt{x^2 - x_0^2}\Big\{3\frac{\alpha'}{A}(1-x) + 2\frac{\beta'}{A}\sqrt{1-x_0^2}\Big\} \\
F_{IP}(x) &= \tfrac{1}{54}\sqrt{x^2 - x_0^2}\Big\{9\xi'\left(-2x + 2 + \sqrt{1-x_0^2}\right) \\
&\qquad + 4\xi(\delta - \tfrac{3}{4})(4x - 4 + \sqrt{1-x_0^2})\Big\} \\
F_{AP}(x) &= \tfrac{1}{6}\Big\{\xi''(2x^2 - x - x_0^2) + 4(\rho - \tfrac{3}{4})\left(4x^2 - 3x - x_0^2\right) \\
&\qquad +2\eta''(1-x)x_0\Big\} \; .
\end{aligned}
$$

For the experimental values of the parameters $\rho$, $\xi$, $\xi'$, $\xi''$, $\delta$, $\eta$, $\eta''$, $\alpha/A$, $\beta/A$, $\alpha'/A$, $\beta'/A$, which are not all independent, see the Data Listings below. Experiments in the past have also been analyzed using the parameters $a$, $b$, $c$, $a'$, $b'$, $c'$, $\alpha/A$, $\beta/A$, $\alpha'/A$, $\beta'/A$ (and $\eta = (\alpha - 2\beta)/2A$), as defined by Kinoshita and Sirlin [5]. They serve as a model-independent summary of all possible measurements on the decay electron (see Listings below). The relations between the two sets of parameters are

$$
\begin{aligned}
\rho - \tfrac{3}{4} &= \tfrac{3}{4}(-a + 2c)/A \; , \\
\eta &= (\alpha - 2\beta)/A \; , \\
\eta'' &= (3\alpha + 2\beta)/A \; , \\
\delta - \tfrac{3}{4} &= \tfrac{9}{4} \cdot \frac{(a' - 2c')/A}{1 - [a + 3a' + 4(b+b') + 6c - 14c']/A} \; , \\
1 - \xi\frac{\delta}{\rho} &= 4\,\frac{[(b+b') + 2(c-c')]/A}{1 - (a - 2c)/A} \; , \\
1 - \xi' &= [(a + a') + 4(b+b') + 6(c+c')]/A \; , \\
1 - \xi'' &= (-2a + 20c)/A \; ,
\end{aligned}
$$

where

$$A \;=\; a \;+\; 4b \;+\; 6c \;.$$

The differential decay probability to obtain a *left-handed* $\nu_e$ with (reduced) energy between $y$ and $y + dy$, neglecting radiative corrections as well as the masses of the electron and of the neutrinos, is given by [6]

$$\frac{d\Gamma}{dy} \;=\; \frac{m_\mu^5\, G_F}{16\pi^3} \;\cdot\; Q_L^{\nu_e} \;\cdot y^2 \Big\{ (1-y) - \omega_L \cdot (y - \tfrac{3}{4}) \Big\} \;.$$

Here, $y = 2\; E_{\nu_e}/m_\mu$. $Q_L^{\nu_e}$ and $\omega_L$ are parameters. $\omega_L$ is the neutrino analog of the spectral shape parameter $\rho$ of Michel. Since in the Standard Model, $Q_L^{\nu_e} = 1,\; \omega_L = 0$, the measurement of $d\Gamma/dy$ has allowed a null-test of the Standard Model (see Listings below).

***Matrix element:*** All results in direct muon decay (energy spectra of the electron and of the neutrinos, polarizations, and angular distributions) and in inverse muon decay (the reaction cross section) at energies well below $m_W c^2$ may be parametrized in terms of amplitudes $g^\gamma_{\varepsilon\mu}$ and the Fermi coupling constant $G_F$, using the matrix element

$$\frac{4G_F}{\sqrt{2}} \sum_{\substack{\gamma=S,V,T \\ \varepsilon,\mu=R,L}} g^\gamma_{\varepsilon\mu} \langle \bar{e}_\varepsilon | \Gamma^\gamma | (\nu_e)_n \rangle \langle \bar{\nu}_\mu)_m | \Gamma_\gamma | \mu_\mu \rangle. \qquad (2)$$

We use the notation of Fetscher *et al.* [2], who in turn use the sign conventions and definitions of Scheck [7]. Here, $\gamma = S, V, T$ indicates a scalar, vector, or tensor interaction; and $\varepsilon, \mu = R, L$ indicate a right- or left-handed chirality of the electron or muon. The chiralities $n$ and $m$ of the $\nu_e$ and $\bar{\nu}_\mu$ are then determined by the values of $\gamma, \varepsilon$ and $\mu$. The particles are represented by fields of definite chirality [8].

As shown by Langacker and London [9], explicit lepton-number nonconservation still leads to a matrix element equivalent to Eq. (2). They conclude that it is not possible, even in principle, to test lepton-number conservation in (leptonic) muon decay if the final neutrinos are massless and are not observed.

The ten complex amplitudes $g^\gamma_{\varepsilon\mu}$ ($g^T_{RR}$ and $g^T_{LL}$ are identically zero) and $G_F$ constitute 19 independent (real) parameters

to be determined by experiment. The Standard Model interaction corresponds to one single amplitude $g^V_{LL}$ being unity and all the others being zero.

The (direct) muon decay experiments are compatible with an arbitrary mix of the scalar and vector amplitudes $g^S_{LL}$ and $g^V_{LL}$ – in the extreme even with purely scalar $g^S_{LL} = 2$, $g^V_{LL} = 0$. The decision in favour of the Standard Model comes from the quantitative observation of inverse muon decay, which would be forbidden for pure $g^S_{LL}$ [2].

***Experimental determination of V−A:*** In order to determine the amplitudes $g^\gamma_{\varepsilon\mu}$ uniquely from experiment, the following set of equations, where the left-hand sides represent experimental results, has to be solved.

$$
\begin{aligned}
a &= 16(|g^V_{RL}|^2 + |g^V_{LR}|^2) + |g^S_{RL} + 6g^T_{RL}|^2 + |g^S_{LR} + 6g^T_{LR}|^2 \\
a' &= 16(|g^V_{RL}|^2 - |g^V_{LR}|^2) + |g^S_{RL} + 6g^T_{RL}|^2 - |g^S_{LR} + 6g^T_{LR}|^2 \\
\alpha &= 8\mathrm{Re}\left\{g^V_{RL}(g^{S*}_{LR} + 6g^{T*}_{LR}) + g^V_{LR}(g^{S*}_{RL} + 6g^{T*}_{RL})\right\} \\
\alpha' &= 8\mathrm{Im}\left\{g^V_{LR}(g^{S*}_{RL} + 6g^{T*}_{RL}) - g^V_{RL}(g^{S*}_{LR} + 6g^{T*}_{LR})\right\} \\
b &= 4(|g^V_{RR}|^2 + |g^V_{LL}|^2) + |g^S_{RR}|^2 + |g^S_{LL}|^2 \\
b' &= 4(|g^V_{RR}|^2 - |g^V_{LL}|^2) + |g^S_{RR}|^2 - |g^S_{LL}|^2 \\
\beta &= -4\mathrm{Re}\left\{g^V_{RR}g^{S*}_{LL} + g^V_{LL}g^{S*}_{RR}\right\} \\
\beta' &= 4\mathrm{Im}\left\{g^V_{RR}g^{S*}_{LL} - g^V_{LL}g^{S*}_{RR}\right\} \\
c &= \tfrac{1}{2}\left\{|g^S_{RL} - 2g^T_{RL}|^2 + |g^S_{LR} - 2g^T_{LR}|^2\right\} \\
c' &= \tfrac{1}{2}\left\{|g^S_{RL} - 2g^T_{RL}|^2 - |g^S_{LR} - 2g^T_{LR}|^2\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
Q^{\nu_e}_L &= 1 - \left\{\tfrac{1}{4}|g^S_{LR}|^2 + \tfrac{1}{4}|g^S_{LL}|^2 + |g^V_{RR}|^2 + |g^V_{RL}|^2 + 3|g^T_{LR}|^2\right\} \\
\omega_L &= \tfrac{3}{4}\,\frac{\{|g^S_{RR}|^2 + 4|g^V_{LR}|^2 + |g^S_{RL} + 2g^T_{RL}|^2\}}{|g^S_{RL}|^2 + |g^S_{RR}|^2 + 4|g^V_{LL}|^2 + 4|g^V_{LR}|^2 + 12|g^T_{RL}|^2\}}\ .
\end{aligned}
$$

It has been noted earlier by C. Jarlskog [10], that certain experiments observing the decay electron are especially informative if the yield the $V$-$A$ values. The complete solution is now found as follows. Fetscher *et al.* [2] introduced four probabilities

$Q_{\varepsilon\mu}(\varepsilon, \mu = R, L)$ for the decay of a $\mu$-handed muon into an $\varepsilon$-handed electron and showed that there exist upper bounds on $Q_{RR}$, $Q_{LR}$, and $Q_{RL}$, and a lower bound on $Q_{LL}$. These probabilities are given in terms of the $g^{\gamma}_{\varepsilon\mu}$'s by

$$Q_{\varepsilon\mu} = \tfrac{1}{4}|g^{S}_{\varepsilon\mu}|^2 + |g^{V}_{\varepsilon\mu}|^2 + 3(1-\delta_{\varepsilon\mu})|g^{T}_{\varepsilon\mu}|^2 \ , \tag{3}$$

where $\delta_{\varepsilon\mu} = 1$ for $\varepsilon = \mu$, and $\delta_{\varepsilon\mu} = 0$ for $\varepsilon \neq \mu$. They are related to the parameters $a$, $b$, $c$, $a', b'$, and $c'$ by

$$\begin{aligned} Q_{RR} &= 2(b+b')/A \ , \\ Q_{LR} &= [(a-a') + 6(c-c')]/2A \ , \\ Q_{RL} &= [(a+a') + 6(c+c')]/2A \ , \\ Q_{LL} &= 2(b-b')/A \ , \end{aligned}$$

with $A = 16$. In the Standard Model, $Q_{LL} = 1$ and the others are zero.

Since the upper bounds on $Q_{RR}$, $Q_{LR}$, and $Q_{RL}$ are found to be small, and since the helicity of the $\nu_\mu$ in pion decay is known from experiment [11,12] to very high precision to be $-1$ [13], the cross section $S$ of *inverse* muon decay, normalized to the $V$-$A$ value, yields [2]

$$|g^{S}_{LL}|^2 \leq 4(1-S) \tag{4}$$

and

$$|g^{V}_{LL}|^2 \ = \ S \ . \tag{5}$$

Thus the Standard Model assumption of a pure $V$-$A$ leptonic charged weak interaction of $e$ and $\mu$ is derived (within errors) from experiments at energies far below mass of the $W^{\pm}$: Eq. (5) gives a lower limit for $V$-$A$, and Eqs. (3) and (4) give upper limits for the other four-fermion interactions. The existence of such upper limits may also be seen from $Q_{RR}+Q_{RL} = (1-\xi')/2$ and $Q_{RR} + Q_{LR} = \frac{1}{2}(1 + \xi/3 - 16\ \xi\delta/9)$. Table 1 gives the current experimental limits on the magnitudes of the $g^{\gamma}_{\varepsilon\mu}$'s.

Limits on the "charge retention" coordinates, as used in the older literature (*e.g.*, Ref. 16), are given by Burkard *et al.* [17].

**Table 1.** Coupling constants $g^{\gamma}_{\varepsilon\mu}$. Ninety-percent confidence level experimental limits. The limits on $|g^S_{LL}|$ and $|g^V_{LL}|$ are from Ref. 14, and the others are from Ref. 15. The experimental uncertainty on the muon polarization in pion decay is included.

| | | |
|---|---|---|
| $\lvert g^S_{RR}\rvert < 0.066$ | $\lvert g^V_{RR}\rvert < 0.033$ | $\lvert g^T_{RR}\rvert \equiv 0$ |
| $\lvert g^S_{LR}\rvert < 0.125$ | $\lvert g^V_{LR}\rvert < 0.060$ | $\lvert g^T_{LR}\rvert < 0.036$ |
| $\lvert g^S_{RL}\rvert < 0.424$ | $\lvert g^V_{RL}\rvert < 0.110$ | $\lvert g^T_{RL}\rvert < 0.122$ |
| $\lvert g^S_{LL}\rvert < 0.550$ | $\lvert g^V_{LL}\rvert > 0.960$ | $\lvert g^T_{LL}\rvert \equiv 0$ |

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