32. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	…
M	M	…
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

1/2×1/2

	$+1/2$	$+1/2$
$+1/2$	$+1$	0
$+1/2$	$-1/2$	$1/2$
$-1/2$	$+1/2$	$1/2$
$-1/2$	$-1/2$	$-1/2$

	1	0
$+1/2$	0	0
$-1/2$	$1/2$	$1/2$
$-1/2$	$1/2$	$-1/2$

	1
-1	$1/2$
$+1$	$1/2$

	0
0	1

	1
-1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1/2

	$+2$	1
$+5/2$	$5/2$	$3/2$
$+1/2$	$3/2$	$-3/2$

	$+2$	$-1/2$
$+2$	$-1/2$	$1/5$
$+1$	$+1/2$	$4/5$
$+1$	$-1/2$	$4/5$
0	$+1/2$	$3/5$
0	$-1/2$	$3/5$
-1	$+1/2$	$2/5$

	$5/2$	$3/2$
$-1/2$	$1/5$	$4/5$
$+1/2$	$4/5$	$-1/5$
$-1/2$	$2/5$	$3/5$
$+1/2$	$3/5$	$-2/5$
$-1/2$	$3/5$	$-2/5$
$+1/2$	$2/5$	$-3/5$

	$5/2$	$3/2$
-1	$-1/2$	$4/5$
-2	$+1/2$	$1/5$
-2	$-1/2$	$1/5$

	$5/2$	$3/2$
-2	$-1/2$	$-4/5$

	$-5/2$
-2	$-1/2$

	1
-2	1

1×1/2

	$+1/2$	$+1/2$
$+1$	$+1$	1
$+1$	$-1/2$	$1/3$
0	$+1/2$	$2/3$
0	$-1/2$	$2/3$
-1	$+1/2$	$1/3$

	$3/2$	$1/2$
$+1/2$	$2/3$	$-1/3$
$-1/2$	$1/3$	$-2/3$

	$3/2$	$-3/2$
$-1/2$	$1/3$	$2/3$

	1
$-1/2$	1

3/2×1/2

	$+3/2$	$+1/2$
$+3/2$	$+1$	$1/4$
$+1/2$	$+1$	$3/4$
$+3/2$	0	$3/4$
$+1/2$	0	$1/4$
$-1/2$	$+1$	$1/4$
$-1/2$	0	$3/4$

	2	1
$+1/2$	$-1/2$	$1/2$
$-1/2$	$+1/2$	$1/2$
$+1/2$	$-1/2$	$1/2$
$-1/2$	$+1/2$	$1/2$
$-3/2$	$+1/2$	1
$-1/2$	$-1/2$	$3/4$

	2	1
$-1/2$	$-1/2$	$3/4$
$-3/2$	$+1/2$	$1/4$

	2
$-3/2$	$-1/2$

	1
$-3/2$	1

2×1

	$+2$	$+1$
$+3$	$+2$	1
$+2$	0	$2/3$
$+1$	$+1$	$2/3$
$+2$	-1	$1/15$
$+1$	0	$8/15$
0	$+1$	$6/15$

	3	2	1
$+2$	$+1$	$1/3$	$-1/3$
$+2$	-1	$1/3$	$-1/3$
$+1$	$+1$	$1/6$	$-1/2$
$+1$	0	$1/2$	$-1/2$
0	$+1$	$1/2$	$-1/2$
$+1$	-1	$1/2$	$1/2$
0	0	0	0
-1	$+1$	$1/2$	$-1/2$

	3	2	1
$+1$	-1	$3/10$	$-3/10$
0	0	$3/5$	$-2/5$
-1	$+1$	$3/10$	$-3/10$

	3	2	1
$+1$	-1	$1/5$	$-1/2$
0	0	$3/5$	$-2/5$
-1	$+1$	$1/5$	$-1/2$

	3	2	1
0	-1	$6/15$	$-1/10$
-1	0	$8/15$	$-1/6$
-2	$+1$	$1/15$	$-1/3$

	3	2	1
-1	-1	$2/3$	$-1/3$
-2	0	$1/3$	$-2/3$

	3
-2	-1

	1
-3	1

1×1

	$+1$	$+1$
$+1$	$+1$	1
$+1$	0	$1/2$
0	$+1$	$1/2$
$+1$	-1	$1/2$
0	0	$1/2$
-1	$+1$	$1/2$

	2	1	0
$+1$	$+1$	$1/2$	$-1/2$
0	$+1$	$1/2$	$-1/2$
$+1$	-1	$1/6$	$1/2$
0	0	$2/3$	0
-1	$+1$	$1/6$	$-1/2$

	2	1
$+1$	-1	$1/3$
0	0	$-1/3$
-1	$+1$	$1/3$

	2
0	-1
-1	0

	-2
-1	$1/2$
-1	$-1/2$

	1
-1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

3/2×3/2

	$+3/2$	$+3/2$
$+3$	$+3/2$	1
$+2$	$+1/2$	$1/2$
$+2$	$+3/2$	$1/2$
$+1$	$+1/2$	$1/2$
$+1$	$+3/2$	$1/2$
0	$+1/2$	$1/2$

	3	2	1
$+2$	$+1/2$	$-1/2$	$3/10$
$+2$	$-1/2$	$-1/2$	$3/10$
$+1$	$+1/2$	$-1/2$	$3/10$
$+1$	$-1/2$	$-1/2$	$3/10$
0	$+1/2$	$-1/2$	$3/10$
0	$-1/2$	$-1/2$	$3/10$

	3	2	1	0
$+3/2$	$-1/2$	$1/5$	$1/2$	$3/10$
$+1/2$	$+1/2$	$3/5$	0	$-2/5$
$-1/2$	$+3/2$	$1/5$	$-1/2$	$3/10$
$+3/2$	$-3/2$	$1/20$	$1/4$	$9/20$
$+1/2$	$-1/2$	$9/20$	$1/4$	$-1/20$
$-1/2$	$+1/2$	$9/20$	$-1/4$	$-1/20$
$-3/2$	$+3/2$	$1/20$	$-1/4$	$9/20$

	4	3	2	1	0
$+2$	$-3/2$	$1/35$	$6/35$	$2/5$	$2/5$
$+1$	$-1/2$	$12/35$	$5/14$	0	$-3/10$
0	$+1/2$	$18/35$	$-3/35$	$-1/5$	$1/5$
-1	$+3/2$	$4/35$	$-27/70$	$2/5$	$-1/10$

	4	3	2	1	0
$+2$	-2	$1/70$	$1/10$	$2/7$	$2/5$	$1/5$
$+1$	-1	$8/35$	$2/5$	$1/14$	$-1/10$	$-1/5$
0	0	$18/35$	0	$-2/7$	0	$1/5$
-1	1	$8/35$	$-2/5$	$1/14$	$1/10$	$-1/5$
-2	2	$1/70$	$-1/10$	$2/7$	$-2/5$	$1/5$

2×3/2

	$+2$	$+3/2$
$+7/2$	$+2$	1
$+5/2$	$+2$	$5/7$
$+5/2$	$+1$	$5/7$
$+3/2$	$+2$	$3/7$
$+3/2$	$+1$	$4/7$
$+1/2$	$+2$	$1/7$
$+1/2$	$+1$	$4/7$
$-1/2$	$+2$	$1/7$
$-1/2$	$+1$	$4/7$
$-3/2$	$+2$	$2/7$

	$7/2$	$5/2$	$3/2$
$+3/2$	$+3/2$	$+3/2$	$+3/2$
$+1/2$	$+3/2$	$-3/7$	$+3/2$
$-1/2$	$+3/2$	$-3/5$	$+3/2$
$-3/2$	$+3/2$	$-18/35$	$1/5$

	$7/2$	$5/2$	$3/2$	$1/2$
$+1/2$	$+1/2$	$+1/2$	$+1/2$
$-1/2$	$+1/2$	$-1/2$	$-1/2$
$-3/2$	$+1/2$	$-1/2$	$-1/2$
$+1/2$	$-3/2$	$1/5$	$1/2$	$3/10$
$-1/2$	$-1/2$	$3/5$	0	$-2/5$
$-3/2$	$+1/2$	$1/5$	$-1/2$	$3/10$

	4	3	2	1	0
$+1$	$-3/2$	$4/35$	$27/70$	$2/5$	$1/10$
0	$-1/2$	$18/35$	$3/35$	$-1/5$	$-1/5$
-1	$1/2$	$12/35$	$-5/14$	0	$3/10$
-2	$3/2$	$1/35$	$-6/35$	$2/5$	$-2/5$

	4	3	2	1
$+1$	-2	$1/14$	$3/10$	$3/7$	$1/5$
0	-1	$3/7$	$1/5$	$-1/14$	$-3/10$
-1	0	$3/7$	$-1/5$	$-1/14$	$3/10$
-2	1	$1/14$	$-3/10$	$3/7$	$-1/5$

	4	3	2
0	-2	$3/14$	$1/2$	$2/7$
-1	-1	$4/7$	0	$-3/7$
-2	0	$3/14$	$-1/2$	$2/7$

	4	3
-1	-2	$1/2$	$1/2$
-2	-1	$1/2$	$-1/2$

	4
-2	-2	1

4×2

	$+2$	$+2$
$+4$	$+3$	1
$+3$	$+3$	$1/2$
$+3$	$+2$	$1/2$
$+2$	$+3$	$1/2$
$+2$	$+2$	$1/2$
$+2$	$+1$	$1/2$
$+1$	$+2$	$1/2$
$+2$	0	$3/14$
$+1$	1	$4/7$
0	2	$3/14$
$+2$	-1	$1/14$
$+1$	0	$3/7$
0	1	$3/7$
-1	2	$1/14$

	4	3	2
$+1$	-2	$1/2$	$-1/2$
$+1$	-1	$1/2$	$-1/2$
$+1$	0	$1/2$	$-1/2$
0	$+1$	$1/2$	$-1/2$
0	0	$1/2$	$-1/2$
0	-1	$1/2$	$-1/2$
-1	$+1$	$1/2$	$-1/2$
-1	0	$1/2$	$-1/2$
-2	$+1$	$1/2$	$-1/2$
$+4$	$+2$	$4/7$	$3/7$	$7/7$
$+3$	$+2$	$3/7$	$4/7$	$-3/7$
$+3$	$+1$	$3/7$	$-1/7$	$-3/7$
$+2$	$+2$	$2/7$	$3/7$	$-4/7$
$+2$	$+1$	$2/7$	$3/7$	$-4/7$
$+2$	0	$2/7$	$-3/7$	$-4/7$
$+1$	$+2$	$1/7$	$4/7$	$-3/7$
$+1$	$+1$	$1/7$	$4/7$	$-3/7$
$+1$	0	$1/7$	$4/7$	$-3/7$
0	$+2$	$1/7$	$3/7$	$-4/7$
0	$+1$	$1/7$	$3/7$	$-4/7$
0	0	$1/7$	$3/7$	$-4/7$
-1	$+2$	$1/7$	$4/7$	$-3/7$
-1	$+1$	$1/7$	$4/7$	$-3/7$
-1	0	$1/7$	$4/7$	$-3/7$
-2	$+2$	$2/7$	$3/7$	$-4/7$
-2	$+1$	$2/7$	$3/7$	$-4/7$
-2	0	$2/7$	$-3/7$	$-4/7$
-3	$+2$	$3/7$	$4/7$	$-3/7$
-3	$+1$	$3/7$	$4/7$	$-3/7$
-4	$+2$	$4/7$	$3/7$	$-4/7$

	4	3	2	1
$+2$	-2	$1/14$	$3/10$	$3/7$	$1/5$
$+2$	-1	$3/7$	$1/5$	$-1/14$	$-3/10$
$+1$	-2	$3/7$	$1/5$	$-1/14$	$-3/10$
$+2$	0	$3/14$	$-1/10$	$3/7$	$-1/5$
$+1$	-1	$3/14$	$-1/10$	$3/7$	$-1/5$
$+1$	0	$3/14$	$-1/10$	$3/7$	$-1/5$
0	-2	$3/14$	$-1/10$	$3/7$	$-1/5$
0	-1	$3/14$	$-1/10$	$3/7$	$-1/5$
0	0	$3/14$	$-1/10$	$3/7$	$-1/5$
-1	-2	$3/14$	$-1/10$	$3/7$	$-1/5$
-1	-1	$3/14$	$-1/10$	$3/7$	$-1/5$
-1	0	$3/14$	$-1/10$	$3/7$	$-1/5$
-2	-2	$1/14$	$-3/10$	$3/7$	$-1/5$
-2	-1	$1/14$	$-3/10$	$3/7$	$-1/5$
-2	0	$1/14$	$-3/10$	$3/7$	$-1/5$

	4	3	2
$+1$	-2	$1/14$	$3/10$	$3/7$	$1/5$
0	-1	$3/7$	$1/5$	$-1/14$	$-3/10$
-1	0	$3/7$	$-1/5$	$-1/14$	$3/10$
-2	1	$1/14$	$-3/10$	$3/7$	$-1/5$

	4	3	2
0	-2	$3/14$	$1/2$	$2/7$
-1	-1	$4/7$	0	$-3/7$
-2	0	$3/14$	$-1/2$	$2/7$

	4	3
-1	-2	$1/2$	$1/2$
-2	-1	$1/2$	$-1/2$

	4
-2	-2	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2}$$

Figure 32.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.