26. KINEMATICS

Revised October 1995.

Throughout this section units are used in which $\hbar = c = 1$. The following conversions are useful: $\hbar c = 197.3$ MeV fermi, $(\hbar c)^2 = 0.3894$ (GeV)^2 mb.

26.1. Lorentz transformations

The energy E and 3-momentum $\boldsymbol{p}$ of a particle of mass m form a 4-vector $p = (E, \boldsymbol{p})$ whose square $p^2 \equiv E^2 - |\boldsymbol{p}|^2 = m^2$. The velocity of the particle is $\boldsymbol{\beta} = \boldsymbol{p}/E$. The energy and momentum $(E^*, \boldsymbol{p}^*)$ viewed from a frame moving with velocity $\boldsymbol{\beta}_f$ are given by

$$\begin{pmatrix} E^* \\ p_{||}^* \end{pmatrix} = \begin{pmatrix} \gamma_f & -\gamma_f\beta_f \\ -\gamma_f\beta_f & \gamma_f \end{pmatrix} \begin{pmatrix} E \\ p_{||} \end{pmatrix} , \quad p_T^* = p_T , \tag{26.1}$$

where $\gamma_f = (1 - \beta_f^2)^{-1/2}$ and p_T $(p_{||})$ are the components of $\boldsymbol{p}$ perpendicular (parallel) to $\boldsymbol{\beta}_f$. The scalar product of two 4-vectors $p_1 \cdot p_2 = E_1 E_2 - \boldsymbol{p}_1 \cdot \boldsymbol{p}_2$ is invariant (frame independent).

In the collision of two particles of masses m_1 and m_2 the total center-of-mass energy can be expressed in the Lorentz-invariant form

$$\begin{aligned} E_{\rm cm} &= \left[(E_1 + E_2)^2 - (\boldsymbol{p}_1 + \boldsymbol{p}_2)^2\right]^{1/2} , \\ &= \left[m_1^2 + m_2^2 + 2E_1E_2(1 - \beta_1\beta_2\cos\theta)\right]^{1/2} , \end{aligned} \tag{26.2}$$

where θ is the angle between the particles. In the frame where one particle (of mass m_2) is at rest (lab frame),

$$E_{\rm cm} = (m_1^2 + m_2^2 + 2E_{1\,\rm lab}\, m_2)^{1/2} . \tag{26.3}$$

The velocity in the lab of the center-of-mass frame is

$$\boldsymbol{\beta}_{\rm cm} = \boldsymbol{p}_{\rm lab}/(E_{1\,\rm lab} + m_2) , \tag{26.4}$$

and

$$\gamma_{\rm cm} = (E_{1\,\rm lab} + m_2)/E_{\rm cm} . \tag{26.5}$$

26.2. Center-of-mass energy and momentum

A beam of particles with mass m and momentum $p_{\rm beam}$ is incident on a fixed target consisting of particles with mass M. The energy of the beam particles $E_{\rm beam}$, the total center-of-mass energy $E_{\rm cm}$, and center of mass momentum of one of the particles $p_{\rm cm}$ are given by

$$E_{\rm beam} = \sqrt{p_{\rm beam}^2 + m^2} \tag{26.6}$$

$$E_{\rm cm} = \sqrt{m^2 + 2E_{\rm beam}M + M^2} \tag{26.7}$$

$$p_{\rm cm} = p_{\rm beam}\frac{M}{E_{\rm cm}} \ . \tag{26.8}$$

For example, if a 0.80 GeV/c kaon beam is incident on a proton target, the center of mass energy is 1.699 GeV and the center of mass momentum of either particle is 0.442 GeV/c. It is also useful to note that

$$E_{\rm cm}\, dE_{\rm cm} = M\, dE_{\rm beam} = M\, \beta_{\rm beam}\, dp_{\rm beam} \ . \tag{26.9}$$

26.3. Lorentz-invariant amplitudes

The invariant amplitude $-i\mathscr{M}$ for a scattering or decay process is determined in perturbation theory by a set of Feynman diagrams. The convention of Bjorken and Drell is used except that fermion spinors are normalized so that $u\overline{u} = 2m$. As an example, the S-matrix for $2 \to 2$ scattering is related to $\mathscr{M}$ by

$$\begin{aligned}\langle p_1' p_2' \, |S| \, p_1 p_2 \rangle &= I - i(2\pi)^4 \, \delta^4(p_1 + p_2 - p_1' - p_2') \\ &\times \frac{\mathscr{M}(p_1,\, p_2;\, p_1',\, p_2')}{(2E_1)^{1/2}\,(2E_2)^{1/2}\,(2E_1')^{1/2}\,(2E_2')^{1/2}} \ .\end{aligned} \tag{26.10}$$

The state normalization is such that

$$\langle p' | p \rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{p}') \ . \tag{26.11}$$

26.4. Particle decays

The partial decay rate of a particle of mass M into n bodies in its rest frame is given in terms of the Lorentz-invariant matrix element $\mathscr{M}$ by

$$d\Gamma = \frac{(2\pi)^4}{2M} \, |\mathscr{M}|^2 \, d\Phi_n\,(P;\, p_1,\, \ldots,\, p_n), \tag{26.12}$$

where $d\Phi_n$ is an element of n-body phase space given by

$$d\Phi_n(P;\, p_1,\, \ldots,\, p_n) = \delta^4\,(P - \sum_{i=1}^n p_i)\, \prod_{i=1}^n \, \frac{d^3p_i}{(2\pi)^3 2E_i} \ . \tag{26.13}$$

This phase space can be generated recursively, viz.

$$d\Phi_n(P;\, p_1,\, \ldots,\, p_n) = d\Phi_j(q;\, p_1,\, \ldots,\, p_j) \\ \times\ d\Phi_{n-j+1}\,(P;\, q,\, p_{i+1},\, \ldots,\, p_n)(2\pi)^3 dq^2\ , \qquad (26.14)$$

where $q^2 = (\sum_{i=1}^{j} E_i)^2 - \left|\sum_{i=1}^{j} \boldsymbol{p}_i\right|^2$. This form is particularly useful in the case where a particle decays into another particle which subsequently decays.

26.4.1. ***Survival probability***: If a particle of mass M has mean proper lifetime τ $(= 1/\Gamma)$ and has momentum $(E, \boldsymbol{p})$, then the probability that it lives for a time t_0 or greater before decaying is given by

$$P(t_0) = e^{-t_0\,\Gamma/\gamma} = e^{-Mt_0\,\Gamma/E}\ , \qquad (26.15)$$

and the probability that it travels a distance x_0 or greater is

$$P(x_0) = e^{-Mx_0\,\Gamma/|\boldsymbol{p}|}\ . \qquad (26.16)$$

26.4.2. ***Two-body decays***:

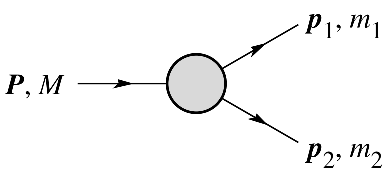

Figure 26.1: Variable definitions for two-body decays.

In the rest frame of a particle of mass M, decaying into 2 particles labeled 1 and 2,

$$E_1 = \frac{M^2 - m_2^2 + m_1^2}{2M}\ , \qquad (26.17)$$

$$|\boldsymbol{p}_1| = |\boldsymbol{p}_2| \\ = \frac{\left[\left(M^2 - (m_1+m_2)^2\right)\left(M^2 - (m_1-m_2)^2\right)\right]^{1/2}}{2M}\ , \qquad (26.18)$$

and

$$d\Gamma = \frac{1}{32\pi^2}\,|\mathscr{M}|^2\,\frac{|\boldsymbol{p}_1|}{M^2}\,d\Omega\ , \qquad (26.19)$$

where $d\Omega = d\phi_1 d(\cos\theta_1)$ is the solid angle of particle 1.

26.4.3. ***Three-body decays***:

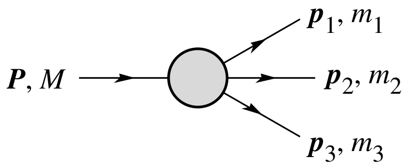

Figure 26.2: Variable definitions for three-body decays.

Defining $p_{ij} = p_i + p_j$, $m_{ij}^2 = p_{ij}^2$, then $m_{12}^2 + m_{23}^2 + m_{13}^2 = M^2 + m_1^2 + m_2^2 + m_3^2$ and $m_{12}^2 = (P - p_3)^2 = M^2 + m_3^2 - 2ME_3$. The relative orientation of the three final-state particles is fixed if their energies are known. Their momenta can therefore be specified by giving three Euler angles (α, β, γ) which specify the orientation of the final system relative to the initial particle. Then

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M} \, |\mathscr{M}|^2 \, dE_1 \, dE_2 \, d\alpha \, d\cos\beta \, d\gamma \; . \tag{26.20}$$

Alternatively

$$d\Gamma = \frac{1}{(2\pi)^5} \frac{1}{16M^2} \, |\mathscr{M}|^2 \, |\boldsymbol{p}_1^*| \, |\boldsymbol{p}_3| \, dm_{12} \, d\Omega_1^* \, d\Omega_3 \; , \tag{26.21}$$

where $(|\boldsymbol{p}_1^*|, \, \Omega_1^*)$ is the momentum of particle 1 in the rest frame of 1 and 2, and Ω_3 is the angle of particle 3 in the rest frame of the decaying particle. $|\boldsymbol{p}_1^*|$ and $|\boldsymbol{p}_3|$ are given by

$$|\boldsymbol{p}_1^*| = \frac{\left[\left(m_{12}^2 - (m_1 + m_2)^2\right)\left(m_{12}^2 - (m_1 - m_2)^2\right)\right]^{1/2}}{2m_{12}} \; , \tag{26.22a}$$

and

$$|\boldsymbol{p}_3| = \frac{\left[\left(M^2 - (m_{12} + m_3)^2\right)\left(M^2 - (m_{12} - m_3)^2\right)\right]^{1/2}}{2M} \; . \tag{26.22b}$$

[Compare with Eq. (26.18).]

Integrating over the angles in Eq. (26.20) (this is only possible if the decaying particle is a scalar or we average over its spin states; otherwise $\mathscr{M}$ depends on α, β, and γ) gives

$$\begin{aligned} d\Gamma &= \frac{1}{(2\pi)^3} \frac{1}{8M} \, |\mathscr{M}|^2 \, dE_1 \, dE_2 \\ &= \frac{1}{(2\pi)^3} \frac{1}{32M^3} \, |\mathscr{M}|^2 \, dm_{12}^2 \, dm_{23}^2 \; . \end{aligned} \tag{26.23}$$

This is the standard form for the Dalitz plot.

26.4.3.1. *Dalitz plot:* If m_{12}^2 is fixed then the range of m_{23}^2 is determined by its values when $\boldsymbol{p}_2$ is parallel or antiparallel to $\boldsymbol{p}_3$.

$$(m_{23}^2)_{\max} =$$

$$(E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} - \sqrt{E_3^{*2} - m_3^2}\right)^2 , \qquad (26.24a)$$

$$(m_{23}^2)_{\min} =$$

$$(E_2^* + E_3^*)^2 - \left(\sqrt{E_2^{*2} - m_2^2} + \sqrt{E_3^{*2} - m_3^2}\right)^2 , \qquad (26.24b)$$

where $E_3^* = (M^2 - m_{12}^2 - m_3^2)/2m_{12}$ and $E_2^* = (m_{12}^2 - m_1^2 + m_2^2)/2m_{12}$ are the energies of particles 2 and 3 in the m_{12} rest frame. The scatter plot in m_{12}^2 and m_{23}^2 has uniform phase space density [see Eq. (26.23)] and is called a Dalitz plot.

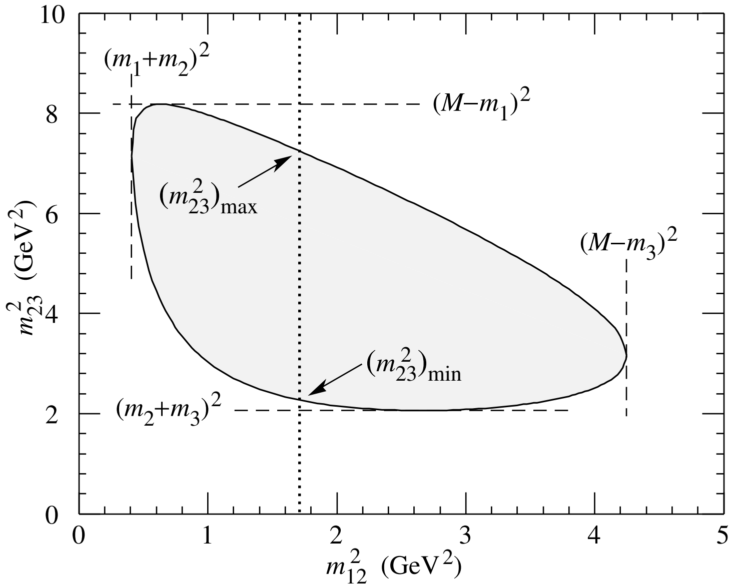

Figure 26.3: Dalitz plot for a three-body final state. In this example, the state is $\pi^+\overline{K}^0 p$ at 3 GeV. Four-momentum conservation restricts events to the shaded region.

A nonuniformity in the plot gives immediate information on $|\mathscr{M}|^2$. For example, in the case of $D \rightarrow K\pi\pi$, bands appear when $m_{(K\pi)} = m_{K^*(892)}$, reflecting the appearance of the decay chain $D \rightarrow K^*(892)\pi \rightarrow K\pi\pi$.

26.4.4. ***Kinematic limits***: In a three-body decay the maximum of $|\boldsymbol{p}_3|$, [given by Eq. (26.22)], is achieved when $m_{12} = m_1 + m_2$, *i.e.*, particles 1 and 2 have the same vector velocity in the rest frame of the decaying particle. If, in addition, $m_3 > m_1, m_2$, then $|\boldsymbol{p}_3|_{\max} > |\boldsymbol{p}_1|_{\max}$, $|\boldsymbol{p}_2|_{\max}$.

26.4.5. ***Multibody decays***: The above results may be generalized to final states containing any number of particles by combining some of the particles into "effective particles" and treating the final states as 2 or 3 "effective particle" states. Thus, if $p_{ijk\ldots} = p_i + p_j + p_k + \ldots$, then

$$m_{ijk\ldots} = \sqrt{p^2{}_{ijk\ldots}}\ , \tag{26.25}$$

and $m_{ijk\ldots}$ may be used in place of *e.g.*, m_{12} in the relations in Sec. 26.4.3 or 26.4.3.1 above.

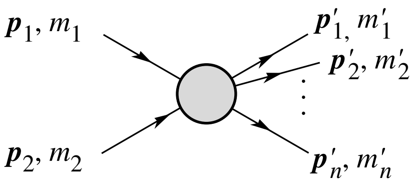

Figure 26.4: Variable definitions for production of an n-body final state.

26.5. Cross sections

The differential cross section is given by

$$d\sigma = \frac{(2\pi)^4 |\mathscr{M}|^2}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \times\ d\Phi_n(p_1 + p_2;\ p_3,\ \ldots,\ p_{n+2})\ . \tag{26.26}$$

[See Eq. (26.13).] In the rest frame of m_2(lab),

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = m_2 p_{1\,\mathrm{lab}}\ ; \tag{26.27a}$$

while in the center-of-mass frame

$$\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} = p_{1\mathrm{cm}}\sqrt{s}\ . \tag{26.27b}$$

26.5.1. ***Two-body reactions***:

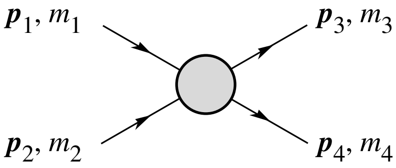

Figure 26.5: Variable definitions for a two-body final state.

Two particles of momenta p_1 and p_2 and masses m_1 and m_2 scatter to particles of momenta p_3 and p_4 and masses m_3 and m_4; the Lorentz-invariant Mandelstam variables are defined by

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ &= m_1^2 + 2E_1E_2 - 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_2 + m_2^2 \ , \end{aligned} \tag{26.28}$$

$$\begin{aligned} t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ &= m_1^2 - 2E_1E_3 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_3 + m_3^2 \ , \end{aligned} \tag{26.29}$$

$$\begin{aligned} u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \\ &= m_1^2 - 2E_1E_4 + 2\boldsymbol{p}_1 \cdot \boldsymbol{p}_4 + m_4^2 \ , \end{aligned} \tag{26.30}$$

and they satisfy

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2 \ . \tag{26.31}$$

The two-body cross section may be written as

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s} \, \frac{1}{|\boldsymbol{p}_{1\text{cm}}|^2} \, |\mathscr{M}|^2 \ . \tag{26.32}$$

In the center-of-mass frame

$$\begin{aligned} t &= (E_{1\text{cm}} - E_{3\text{cm}})^2 - (p_{1\text{cm}} - p_{3\text{cm}})^2 \\ &\qquad - 4p_{1\text{cm}}\, p_{3\text{cm}} \sin^2(\theta_{\text{cm}}/2) \\ &= t_0 - 4p_{1\text{cm}}\, p_{3\text{cm}} \sin^2(\theta_{\text{cm}}/2) \ , \end{aligned} \tag{26.33}$$

where θ_{cm} is the angle between particle 1 and 3. The limiting values t_0 ($\theta_{\text{cm}} = 0$) and t_1 ($\theta_{\text{cm}} = \pi$) for $2 \to 2$ scattering are

$$\begin{aligned} t_0(t_1) = {} & \left[\frac{m_1^2 - m_3^2 - m_2^2 + m_4^2}{2\sqrt{s}} \right]^2 \\ & - \left\{ \left[\left(\frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \right)^2 - m_1^2 \right]^{1/2} \right. \\ & \left. \mp \left[\left(\frac{s + m_3^2 - m_4^2}{2\sqrt{s}} \right)^2 - m_3^2 \right]^{1/2} \right\}^2 \ . \end{aligned} \tag{26.34}$$

In the literature the notation $t_{\rm min}$ ($t_{\rm max}$) for t_0 (t_1) is sometimes used. This usage should be discouraged since $t_0 > t_1$. The center-of-mass energies and momenta of the incoming particles are

$$E_{1\rm cm} = \frac{s + m_1^2 - m_2^2}{2\sqrt{s}} \ , \qquad E_{2\rm cm} = \frac{s + m_2^2 - m_1^2}{2\sqrt{s}} \ , \tag{26.35}$$

$$\begin{aligned} p_{\rm cm} &= \frac{\left[\left(s-(m_1+m_2)^2\right)\left(s-(m_1-m_2)^2\right)\right]^{1/2}}{2\sqrt{s}} \\ &= \frac{p_{1\,\rm lab} m_2}{\sqrt{s}} \ , \end{aligned} \tag{26.36}$$

Here the subscript lab refers to the frame where particle 2 is at rest. [For other relations see Eqs. (26.2)–(26.4).]

26.5.2. ***Inclusive reactions***: Choose some direction (usually the beam direction) for the z-axis; then the energy and momentum of a particle can be written as

$$E = m_T \cosh y \ , \ p_x \ , \ p_y \ , \ p_z = m_T \sinh y \ , \tag{26.37}$$

where m_T is the transverse mass

$$m_T^2 = m^2 + p_x^2 + p_y^2 \ , \tag{26.38}$$

and the rapidity y is defined by

$$\begin{aligned} y &= \frac{1}{2} \ln \left(\frac{E + p_z}{E - p_z} \right) \\ &= \ln \left(\frac{E + p_z}{m_T} \right) = \tanh^{-1} \left(\frac{p_z}{E} \right) \ . \end{aligned} \tag{26.39}$$

Under a boost in the z-direction to a frame with velocity β, $y \rightarrow y + \tanh^{-1}\beta$. Hence the shape of the rapidity distribution dN/dy is invariant. The invariant cross section may also be rewritten

$$E\frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{d\phi\, dy\, p_T\, dp_T} \Longrightarrow \frac{d^2\sigma}{\pi\, dy\, d(p_T^2)} \ . \tag{26.40}$$

The second form is obtained using the identity $dy/dp_z = 1/E$, and the third form represents the average over ϕ.

Feynman's x variable is given by

$$x = \frac{p_z}{p_{z\,\rm max}} \approx \frac{E + p_z}{(E + p_z)_{\rm max}} \ ; \tag{26.41}$$

in the center-of-mass frame,

$$x \approx \frac{2p_{z\,\rm cm}}{\sqrt{s}} \approx \frac{2m_T \sinh y_{\rm cm}}{\sqrt{s}} \ . \tag{26.42}$$

For $y_{\rm cm}$ such that $e^{-2y_{\rm cm}} \ll 1$,

$$x \approx \frac{m_T}{\sqrt{s}} e^{y_{\rm cm}} \tag{26.43}$$

and

$$(y_{\rm cm})_{\rm max} = \ln \left(\sqrt{s}/m\right) \ . \tag{26.44}$$

The definition of rapidity [Eq. (26.39)] may be expanded to obtain

$$\begin{aligned} y &= \frac{1}{2} \ln \frac{\cos^2(\theta/2) + m^2/4p^2 + \ldots}{\sin^2(\theta/2) + m^2/4p^2 + \ldots} \\ &\approx -\ln \tan(\theta/2) \equiv \eta \end{aligned} \tag{26.45}$$

if the particle has zenith angle θ. The pseudorapidity η defined by the second line is approximately equal to the rapidity y for $p \gg m$ and $\theta \gg 1/\gamma$, and in any case can be measured when the mass and momentum of the particle is unknown. From the definition one can obtain the identities

$$\sinh \eta = \cot \theta \tag{26.46}$$

$$\cosh \eta = 1/\sin \theta \tag{26.47}$$

$$\tanh \eta = \cos \theta \ . \tag{26.48}$$

26.5.3. ***Partial waves***: The amplitude in the center of mass for elastic scattering of spinless particles may be expanded in Legendre polynomials

$$f(k,\theta) = \frac{1}{k} \sum_\ell (2\ell+1) a_\ell P_\ell(\cos\theta) \ , \tag{26.49}$$

where k is the c.m. momentum, θ is the c.m. scattering angle, $a_\ell = (\eta_\ell e^{2i\delta_\ell} - 1)/2i$, $0 \le \eta_\ell \le 1$, and δ_ℓ is the phase shift of the ℓ^{th} partial wave. For purely elastic scattering, $\eta_\ell = 1$. The differential cross section is

$$\frac{d\sigma}{d\Omega} = |f(k,\theta)|^2 \ . \tag{26.50}$$

The optical theorem states that

$$\sigma_{\rm tot} = \frac{4\pi}{k} {\rm Im}\, f(k,0) \ , \tag{26.51}$$

and the cross section in the ℓ^{th} partial wave is therefore bounded:

$$\sigma_\ell = \frac{4\pi}{k^2}(2\ell+1)|a_\ell|^2 \le \frac{4\pi(2\ell+1)}{k^2} \ . \tag{26.52}$$

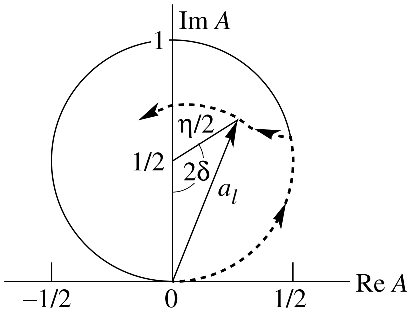

Figure 26.6: Argand plot for the display of a partial-wave amplitude as a function of energy.

The evolution with energy of partial-wave amplitude a_ℓ can be displayed as a trajectory in an Argand plot, as shown in Fig. 26.6.

The usual Lorentz-invariant matrix element $\mathscr{M}$ (see Sec. 26.3 above) for the elastic process is related to $f(k,\theta)$ by

$$\mathscr{M} = -8\pi\sqrt{s}\, f(k,\theta) \; , \tag{26.53}$$

so

$$\sigma_{\rm tot} = -\frac{1}{2k\sqrt{s}}\,\mathrm{Im}\,\mathscr{M}(t=0) \; , \tag{26.54}$$

where s and t are the center-of-mass energy squared and momentum transfer squared, respectively (see Sec. 26.4.1).

26.5.3.1. *Resonances:* The Breit-Wigner form for an elastic amplitude a_ℓ with a resonance at c.m. energy E_R, elastic width $\Gamma_{\rm el}$, and total width $\Gamma_{\rm tot}$ is

$$a_\ell = \frac{\Gamma_{\rm el}/2}{E_R - E - i\Gamma_{\rm tot}/2} \; , \tag{26.55}$$

where E is the c.m. energy. As shown in Fig. 26.7, in the absence of background the elastic amplitude traces a counterclockwise circle with center $ix_{\rm el}/2$ and radius $x_{\rm el}/2$, where the elasticity $x_{\rm el} = \Gamma_{\rm el}/\Gamma_{\rm tot}$. The amplitude has a pole at $E = E_R - i\Gamma_{\rm tot}/2$.

The Breit-Wigner cross section for a spin-J resonance produced in the collision of particles of spin S_1 and S_2 is

$$\sigma_{BW}(E) = \frac{(2J+1)}{(2S_1+1)(2S_2+1)}\,\frac{\pi}{k^2}\,\frac{B_{\rm in}B_{\rm out}\Gamma^2_{\rm tot}}{(E-E_R)^2+\Gamma^2_{\rm tot}/4} \; , \tag{26.56}$$

where k is the c.m. momentum, E is the c.m. energy, and $B_{\rm in}$ and $B_{\rm out}$ are the branching fractions of the resonance into the entrance and exit channels. The $2S+1$ factors are the multiplicities of the incident spin states, so they are replaced by 2 for photons, *etc.* This expression is valid only for a particle of narrow width. If the width

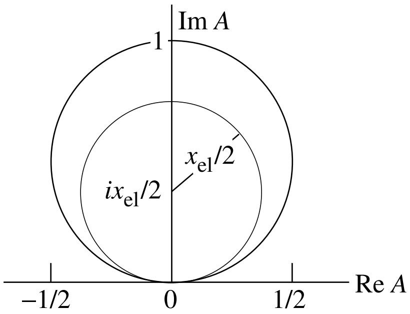

Figure 26.7: Argand plot for a resonance.

is not small, Γ_{tot} cannot be treated as a constant independent of E. There are many other forms for σ_{BW}, all of which are equivalent to the one given here in the narrow-width case. Some of these forms may be more appropriate if the resonance is broad.

The relativistic Breit-Wigner form corresponding to Eq. (26.55) is:

$$a_\ell = \frac{-m\Gamma_{el}}{s - m^2 + im\Gamma_{tot}} \ . \tag{26.57}$$

A better form incorporates the known kinematic dependences, replacing $m\Gamma_{tot}$ by $\sqrt{s}\,\Gamma_{tot}(s)$, where $\Gamma_{tot}(s)$ is the width the resonance particle would have if its mass were $\sqrt{s}$, and correspondingly $m\Gamma_{el}$ by $\sqrt{s}\,\Gamma_{el}(s)$ where $\Gamma_{el}(s)$ is the partial width in the incident channel for a mass $\sqrt{s}$:

$$a_\ell = \frac{-\sqrt{s}\,\Gamma_{el}(s)}{s - m^2 + i\sqrt{s}\,\Gamma_{tot}(s)} \ . \tag{26.58}$$

For the Z boson, all the decays are to particles whose masses are small enough to be ignored, so on dimensional grounds $\Gamma_{tot}(s) = \sqrt{s}\,\Gamma_0/m_Z$, where Γ_0 defines the width of the Z, and $\Gamma_{el}(s)/\Gamma_{tot}(s)$ is constant. A full treatment of the line shape requires consideration of dynamics, not just kinematics. For the Z this is done by calculating the radiative corrections in the Standard Model.