# 27. CROSS-SECTION FORMULAE FOR SPECIFIC PROCESSES

## 27.1. Leptoproduction

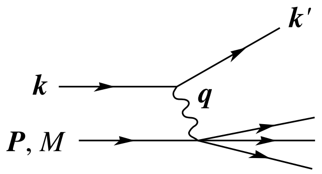


**Figure 27.1:** Kinematic quantities for description of lepton-nucleon scattering. $k$ and $k'$ are the four-momenta of incoming and outgoing leptons, $P$ is the four-momentum of a nucleon with mass $M$. The exchanged particle is a $\gamma$, $W^{\pm}$, or $Z^0$; it transfers four-momentum $q = k - k'$ to the target.

Invariant quantities:

$\nu = \dfrac{q \cdot P}{M} = E - E'$ is the lepton's energy loss in the lab (in earlier literature sometimes $\nu = q \cdot P$). Here, $E$ and $E'$ are the initial and final lepton energies in the lab.

$Q^2 = -q^2 = 2(EE' - \overrightarrow{k} \cdot \overrightarrow{k}') - m_\ell^2 - m_{\ell'}^2$ where $m_\ell(m_{\ell'})$ is the initial (final) lepton mass. If $EE' \sin^2(\theta/2) \gg m_\ell^2,\ m_{\ell'}^2$, then

$\approx 4EE' \sin^2(\theta/2)$, where $\theta$ is the lepton's scattering angle in the lab.

$x = \dfrac{Q^2}{2M\nu}$ In the parton model, $x$ is the fraction of the target nucleon's momentum carried by the struck quark. [See section on Quantum Chromodynamics (Sec. 28 of this *Review*.]

$y = \dfrac{q \cdot P}{k \cdot P} = \dfrac{\nu}{E}$ is the fraction of the lepton's energy lost in the lab.

$W^2 = (P + q)^2 = M^2 + 2M\nu - Q^2$ is the mass squared of the system recoiling against the lepton.

$$s = (k + P)^2 = \frac{Q^2}{xy} + M^2$$

**27.1.1. *Leptoproduction cross sections*:**

$$\frac{d^2\sigma}{dx\,dy} = \nu\,(s-M^2)\,\frac{d^2\sigma}{d\nu\,dQ^2} = \frac{2\pi\,M\nu}{E'}\frac{d^2\sigma}{d\Omega_{\rm lab}\,dE'}$$

$$= x(s-M^2)\,\frac{d^2\sigma}{dx\,dQ^2}\ . \qquad (27.1)$$

**27.1.2. *Electroproduction structure functions*:** The neutral-current process, $eN \to eX$, is parity conserving at low $Q^2$ and can be written in terms of two structure functions $F_1^{\rm NC}(x,Q^2)$ and $F_2^{\rm NC}(x,Q^2)$:

$$\frac{d^2\sigma}{dx\,dy} = \frac{4\pi\,\alpha^2(s-M^2)}{Q^4}$$

$$\times\left[(1-y)\,F_2^{\rm NC} + y^2\,xF_1^{\rm NC} - \frac{M^2}{(s-M^2)}\,xy\,F_2^{\rm NC}\right]\ . \qquad (27.2)$$

The charged-current processes, $e^-N \to \nu X$, $\nu N \to e^-X$, and $\overline{\nu}N \to e^+X$, are parity violating and can be written in terms of three structure functions $F_1^{\rm CC}(x,Q^2)$, $F_2^{\rm CC}(x,Q^2)$, and $F_3^{\rm CC}(x,Q^2)$:

$$\frac{d^2\sigma}{dx\,dy} = \frac{G_F^2\,(s-M^2)}{2\pi}\frac{M_W^4}{(Q^2+M_W^2)^2} \qquad (27.3)$$

$$\times\ \left\{\left[1-y-\frac{M^2xy}{(s-M^2)}\right]F_2^{\rm CC} + \frac{y^2}{2}\,2x\,F_1^{\rm CC} \pm (y-\frac{y^2}{2})\,x\,F_3^{\rm CC}\right\}\ ,$$

where the last term is positive for the $e^-$ and $\nu$ reactions and negative for $\overline{\nu}N \to e^+X$.

**27.1.3. *The QCD parton model*:** In the QCD parton model, the structure functions defined above can be expressed in terms of parton distribution functions. The quantity $f_i(x,Q^2)dx$ is the probability that a parton of type $i$ (quark, antiquark, or gluon), carries a momentum fraction between $x$ and $x+dx$ of the nucleon's momentum in a frame where the nucleon's momentum is large. For the cross section corresponding to the *neutral-current process* $ep \to eX$, we have for $s \gg M^2$ (in the case where the incoming electron is either left- ($L$) or right- ($R$) handed):

$$\frac{d^2\sigma}{dx\,dy} = \frac{\pi\alpha^2}{sx^2\,y^2}\left[\sum_q (x\,f_q\,(x,\,Q^2) + x\,f_{\overline{q}}\,(x,\,Q^2)\right]$$

$$\times\left[A_q + (1-y)^2\,B_q\right]\ . \qquad (27.4)$$

Here the index $q$ refers to a quark flavor (*i.e.*, $u$, $d$, $s$, $c$, $b$, or $t$), and

$$A_q = \left(-q_q + g_{Lq}\,g_{Le}\,\frac{Q^2}{Q^2+M_Z^2}\right)^2 + \left(-q_q + g_{Rq}\,g_{Re}\,\frac{Q^2}{Q^2+M_Z^2}\right)^2,$$

$$\tag{27.5}$$

$$B_q = \left(-q_q + g_{Rq}\, g_{Le}\, \frac{Q^2}{Q^2+M_Z^2}\right)^2 + \left(-q_q + g_{Lq}\, g_{Re}\, \frac{Q^2}{Q^2+M_Z^2}\right)^2 . \tag{27.6}$$

Here $q_q$ is the charge of flavor $q$. For a left-handed electron, $g_{Re} = 0$ and $g_{Le} = (-1/2 + \sin^2\theta_W)/(\sin\theta_W \cos\theta_W)$, while for a right-handed one, $g_{Le} = 0$ and $g_{Re} = (\sin^2\theta_W)/(\sin\theta_W \cos\theta_W)$. For the quarks, $g_{Lq} = (T_3 - q_q \sin^2\theta_W)/(\sin\theta_W \cos\theta_W)$, and $g_{Rq} = (-q_q \sin^2\theta_W)$ / $(\sin\theta_W \cos\theta_W)$.

For neutral-current *neutrino (antineutrino) scattering*, the same formula applies with $g_{Le}$ replaced by $g_{L\nu} = 1/(2\sin\theta_W \cos\theta_W)$ ($g_{L\overline{\nu}} = 0$) and $g_{Re}$ replaced by $g_{R\nu} = 0$ $[g_{R\overline{\nu}} = -1/(2\sin\theta_W \cos\theta_W)]$.

In the case of the *charged-current processes* $e^-_L p \to \nu X$ and $\overline{\nu} p \to e^+ X$, Eq. (27.3) applies with

$$F_2 = 2xF_1 = 2x\Big[f_u(x,Q^2) + f_c(x,Q^2)$$

$$+\ f_t(x,Q^2) + f_{\overline{d}}\,(x,Q^2) + f_{\overline{s}}\,(x,Q^2) + f_{\overline{b}}\,(x,Q^2)\Big]\ , \tag{27.7}$$

$$F_3 = 2\Big[f_u(x,Q^2) + f_c(x,Q^2)$$

$$+\ f_t(x,Q^2) - f_{\overline{d}}\,(x,Q^2) - f_{\overline{s}}\,(x,Q^2) - f_{\overline{b}}\,(x,Q^2)\Big]\ . \tag{27.8}$$

For the process $\nu p \to e^- X$:

$$F_2 = 2xF_1 = 2x\Big[f_d(x,Q^2) + f_s(x,Q^2)$$

$$+\ f_b(x,Q^2) + f_{\overline{u}}\,(x,Q^2) + f_{\overline{c}}\,(x,Q^2) + f_{\overline{t}}\,(x,Q^2)\Big]\ , \tag{27.9}$$

$$F_3 = 2\Big[f_d(x,Q^2) + f_s(x,Q^2)$$

$$+\ f_b(x,Q^2) - f_{\overline{u}}\,(x,Q^2) - f_{\overline{c}}\,(x,Q^2) - f_{\overline{t}}\,(x,Q^2)\Big]\ . \tag{27.10}$$

## 27.2. $e^+e^-$ annihilation

For pointlike spin-1/2 fermions in the c.m., the differential cross section for $e^+e^- \to f\overline{f}$ via single photon annihilation is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\,\beta\ \Big[1 + \cos^2\theta + (1-\beta^2)\sin^2\theta\Big]\, Q_f^2\ , \tag{27.11}$$

where $\beta$ is the velocity of the final state fermion in the center of mass and $Q_f$ is the charge of the fermion in units of the proton charge. For

$\beta \to 1$,

$$\sigma = \frac{4\pi\alpha^2}{3s}\, Q_f^2 = \frac{86.8\, Q_f^2\, nb}{s(\mathrm{GeV}^2)}\ . \tag{27.12}$$

At higher energies the $Z^0$ (mass $M_Z$ and width $\Gamma_Z$) must be included, and the differential cross section for $e^+e^- \to f\overline{f}$ becomes

$$\begin{aligned}\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\,\beta\,\Big[ & Q_f^2[1+\cos^2\theta + (1-\beta^2)\,\sin^2\theta] \\ & - 2Q_f\,\chi_1\Big\{VV_f\,[1+\cos^2\theta+(1-\beta^2)\,\sin^2\theta] - 2a_f\,\beta\cos\theta\Big\} \\ & + \chi_2\Big\{V_f^2\,(1+V^2)\,[1+\cos^2\theta+(1-\beta^2)\,\sin^2\theta] \\ & + \beta^2\,a_f^2(1+V^2)\,[1+\cos^2\theta] - 8\beta\,VV_f\,a_f\cos\theta\Big\}\Big]\ ,\end{aligned} \tag{27.13}$$

$$\chi_1 = \frac{1}{16\sin^2\theta_W\cos^2\theta_W}\ \frac{s(s-M_Z^2)}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}\ , \tag{27.14}$$

$$\chi_2 = \frac{1}{256\sin^4\theta_W\cos^4\theta_W}\ \frac{s^2}{(s-M_Z^2)^2+\Gamma_Z^2 M_Z^2}\ , \tag{27.15}$$

$$V = -1 + 4\sin^2\theta_W\ , \tag{27.16}$$

$$a_f = 2T_{3f}\ , \tag{27.17}$$

$$V_f = 2T_{3f} - 4Q_f\sin^2\theta_W\ , \tag{27.18}$$

where the subscript $f$ refers to the particular fermion and

$$T_3 = +1/2 \quad \text{for } \nu_e,\, \nu_\mu,\, \nu_\tau,\, u,\, c,\, t, \tag{27.19$a$}$$

$$T_3 = -1/2 \quad \text{for } e^-,\, \mu^-,\, \tau^-,\, d,\, s,\, b. \tag{27.19$b$}$$

## 27.3. Two-photon process at $e^+e^-$ colliders

When an $e^+$ and an $e^-$ collide with energies $E_1$ and $E_2$, they emit $dn_1$ and $dn_2$ virtual photons with energies $\omega_1$ and $\omega_2$ and 4-momenta $q_1$ and $q_2$. In the equivalent photon approximation, the cross section for $e^+e^- \to e^+e^-X$ is related to the cross section for $\gamma\gamma \to X$ by (Ref. 1)

$$d\sigma_{e^+e^-\to e^+e^-X}\,(s) = dn_1\,dn_2\,d\sigma_{\gamma\gamma\to X}\,(W^2) \tag{27.20}$$

where $s = 4E_1E_2$, $W^2 = 4\omega_1\omega_2$ and

$$dn_i = \frac{\alpha}{\pi}\left[1 - \frac{\omega_i}{E_i} + \frac{\omega_i^2}{2E_i^2} - \frac{m_e^2\omega_i^2}{(-q_i^2)E_i^2}\right]\frac{d\omega_i}{\omega_i}\,\frac{d(-q_i^2)}{(-q_i^2)}\ . \tag{27.21}$$

After integration (including that over $q_i^2$ in the region $m_e^2\omega_i^2/E_i(E_i-\omega_i) \le -q_i^2 \le (-q^2)_{\max}$), the cross section is

$$\sigma_{e^+e^-\to e^+e^-X}(s) = \frac{\alpha^2}{\pi^2}\int_{z_{th}}^{1}\frac{dz}{z}\left[f(z)\left(\ln\frac{(-q^2)_{\max}}{m_e^2 z}-1\right)^2 - \tfrac{1}{3}\left(\ln\frac{1}{z}\right)^3\right]\sigma_{\gamma\gamma\to X}(zs)\ ;$$

$$f(z) = \left(1+\tfrac{1}{2}z\right)^2\ln\frac{1}{z} - \tfrac{1}{2}(1-z)(3+z)\ ;$$

$$z = \frac{W^2}{s}\ . \tag{27.22}$$

The quantity $(-q^2)_{\max}$ depends on properties of the produced system $X$, in particular, $(-q^2)_{\max} \sim m_\rho^2$ for hadron production ($X=h$) and $(-q^2)_{\max} \sim W^2$ for lepton pair production ($X=\ell^+\ell^-$, $\ell = e,\ \mu,\ \tau$).

For production of a resonance of mass $m_R$ and spin $J \neq 1$

$$\sigma_{e^+e^-\to e^+e^-R}(s) = (2J+1)\frac{8\alpha^2\Gamma_{R\to\gamma\gamma}}{m_R^3}$$

$$\times\left[f(m_R^2/s)\left(\ln\frac{sm_V^2}{m_e^2m_R^2}-1\right)^2 - \frac{1}{3}\left(\ln\frac{s}{m_R^2}\right)^3\right] \tag{27.23}$$

where $m_V$ is the mass that enters into the form factor of the $\gamma\gamma\to R$ transition: $m_V \sim m_\rho$ for $R = \pi^0,\ \rho^0,\ \omega,\ \phi,\ \ldots$, $m_V \sim m_R$ for $R = c\overline{c}$ or $b\overline{b}$ resonances.

## 27.4. Inclusive hadronic reactions

One-particle inclusive cross sections $Ed^3\sigma/d^3p$ for the production of a particle of momentum $p$ are conveniently expressed in terms of rapidity (see above) and the momentum $p_T$ transverse to the beam direction (defined in the center-of-mass frame)

$$E\frac{d^3\sigma}{d^3p} = \frac{d^3\sigma}{d\phi\, dy\, p_T\, dp_T}\ . \tag{27.24}$$

In the case of processes where $p_T$ is large or the mass of the produced particle is large (here large means greater than 10 GeV), the parton model can be used to calculate the rate. Symbolically

$$\sigma_{\text{hadronic}} = \sum_{ij}\int f_i(x_1,\, Q^2)\, f_j(x_2,\, Q^2)\, dx_1\, dx_2\, \widehat{\sigma}_{\text{partonic}}\ , \tag{27.25}$$

where $f_i(x,\, Q^2)$ is the parton distribution introduced above and $Q$ is a typical momentum transfer in the partonic process and $\widehat{\sigma}$ is the partonic cross section. Two examples will help to clarify. The production of a $W^+$ in $pp$ reactions at rapidity $y$ in the center-of-mass frame is given by

$$\begin{aligned}\frac{d\sigma}{dy} &= \frac{G_F\ \pi\sqrt{2}}{3} \\ &\quad\times \tau\Bigg[\cos^2\theta_c\Big(u(x_1\ ,\ M_W^2)\ \overline{d}\ (x_2, M_W^2) \\ &\qquad\qquad + u(x_2\ ,\ M_W^2)\ \overline{d}\ (x_1, M_W^2)\Big) \\ &\qquad + \sin^2\theta_c\Big(u(x_1\ ,\ M_W^2)\ \overline{s}\ (x_2\ ,\ M_W^2) \\ &\qquad\qquad + s(x_2, M_W^2)\ \overline{u}\ (x_1, M_W^2)\Big)\Bigg]\ , \end{aligned} \tag{27.26}$$

where $x_1 = \sqrt{\tau}\ e^y$, $x_2 = \sqrt{\tau}\ e^{-y}$, and $\tau = M_W^2/s$. Similarly the production of a jet in $pp$ (or $p\overline{p}$) collisions is given by

$$\begin{aligned}\frac{d^3\sigma}{d^2p_T\ dy} &= \sum_{ij}\int f_i(x_1\ ,\ p_T^2)\ f_j(x_2, p_T^2) \\ &\quad\times \left[\widehat{s}\ \frac{d\widehat{\sigma}}{d\widehat{t}}\right]_{ij} dx_1\ dx_2\ \delta(\widehat{s} + \widehat{t} + \widehat{u})\ , \end{aligned} \tag{27.27}$$

where the summation is over quarks, gluons, and antiquarks. Here

$$s = (p_1 + p_2)^2\ , \tag{27.28}$$

$$t = (p_1 - p_{\text{jet}})^2\ , \tag{27.29}$$

$$u = (p_2 - p_{\text{jet}})^2\ , \tag{27.30}$$

$p_1$ and $p_2$ are the momenta of the incoming $p$ and $p$ (or $\overline{p}$) and $\widehat{s}$, $\widehat{t}$, and $\widehat{u}$ are $s$, $t$, and $u$ with $p_1 \to x_1 p_1$ and $p_2 \to x_2\, p_2$. The partonic cross section $\widehat{s}\,[(d\widehat{\sigma})/(d\widehat{t})]$ can be found in Ref. 2. Example: for the process $gg \to q\overline{q}$,

$$\widehat{s}\ \frac{d\sigma}{dt} = 3\alpha_s^2\ \frac{(\widehat{t}^{\,2} + \widehat{u}^2)}{8\widehat{s}}\ \left[\frac{4}{9\,\widehat{t}\,\widehat{u}} - \frac{1}{\widehat{s}^{\,2}}\right]\ . \tag{27.31}$$

The prediction of Eq. (27.27) is compared to data from the UA1 and UA2 collaborations in Fig. 36.7 in the Plots of Cross Sections and Related Quantities section of this *Review.*

## 27.5. One-particle inclusive distributions

In order to describe one-particle inclusive production in $e^+e^-$ annihilation or deep inelastic scattering, it is convenient to introduce a fragmentation function $D_i^h\,(z,Q^2)/z$ which is the probability that a parton of type $i$ and momentum $p$ will fragment into a hadron of type $h$ and momentum $zp$. The $Q^2$ evolution is predicted by QCD and is similar to that of the parton distribution functions [see section on Quantum Chromodynamics (Sec. 28 of this *Review*)]. The $D_i^h(z,Q^2)$ are normalized so that

$$\sum_h \int D_i^h\,(z,Q^2)dz = 1\ . \tag{27.32}$$

If the contributions of the $Z$ boson and three-jet events are neglected, the cross section for producing a hadron $h$ in $e^+e^-$ annihilation is given by

$$\frac{1}{\sigma_{\rm had}}\frac{d\sigma}{dz} = \frac{\sum_i e_i^2\ D_i^h\,(z,Q^2)}{\sum_i e_i^2}\ , \tag{27.33}$$

where $e_i$ is the charge of quark-type $i$, $\sigma_{\rm had}$ is the total hadronic cross section, and the momentum of the hadron is $zE_{\rm cm}/2$.

In the case of deep inelastic muon scattering, the cross section for producing a hadron of energy $E_h$ is given by

$$\frac{1}{\sigma_{\rm tot}}\ \frac{d\sigma}{dz} = \frac{\sum_i e_i^2\ q_i(x,Q^2)\ D_i^h(z,Q^2)}{\sum_i e_i^2\ q_i(x,Q^2)}\ , \tag{27.34}$$

where $E_h = \nu z$. (For the kinematics of deep inelastic scattering, see Sec. 26.4.2 of the Kinematics section of this *Review.*) The fragmentation functions for light and heavy quarks have a different $z$ dependence; the former peak near $z = 0$. They are illustrated in Fig. 36.13 in the section on Plots of Cross Sections and Related Quantities (Sec. 36 of this *Review*).

**References:**

1. V.M. Budnev, I.F. Ginzburg, G.V. Meledin, and V.G. Serbo,Phys. Reports **15C**, 181 (1975);

   See also S. Brodsky, T. Kinoshita, and H. Terazawa, Phys. Rev. **D4**, 1532 (1971).

2. G.F. Owens, F. Reya, and M. Glück, Phys. Rev. **D18**, 1501 (1978).