23. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

Notation:

J	J	...
M	M	...
m_1	m_2	
m_1	m_2	Coefficients
.	.	
.	.	
.	.	

$1/2\times1/2$

	1	
$+1/2\ +1/2$	1	$1\quad 0$
		$0\quad 0$

	$+1/2\ -1/2$	$-1/2\ +1/2$	1
$+1/2\ +1/2$	$1/2$	$1/2$	1
$-1/2\ -1/2$	$1/2$	$-1/2$	-1

	$-1/2\ -1/2$	1
$+1/2\ +1/2$	1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$2\times1/2$

	$5/2$	
$+2\ \ 1/2$	$+5/2$	$5/2\ \ 3/2$
	1	$3/2\ +3/2$

	$+2\ -1/2$	$+1\ +1/2$	$5/2\ \ 3/2$
$+2\ \ 1/2$	$1/5$	$4/5$	$+1/2\ +1/2$
$+1\ \ 1/2$	$4/5$	$-1/5$	$+1/2\ +1/2$

	$+1\ -1/2$	$0\ +1/2$	$5/2\ \ 3/2$
$+1\ \ 1/2$	$2/5$	$3/5$	$-1/2\ -1/2$
$0\ \ 1/2$	$3/5$	$-2/5$	$-1/2\ -1/2$

	$0\ -1/2$	$-1\ +1/2$	$5/2\ \ 3/2$
$0\ \ 1/2$	$3/5$	$2/5$	$-3/2\ -3/2$
$-1\ \ 1/2$	$2/5$	$-3/5$	$-3/2\ -3/2$

	$-1\ -1/2$	$-2\ +1/2$	$5/2$
$-1\ \ 1/2$	$4/5$	$1/5$	$-5/2$
$-2\ \ 1/2$	$1/5$	$-4/5$	$-5/2$

	$-2\ -1/2$	1
$-2\ \ 1/2$	1	1

$1\times1/2$

	$3/2$	
$+1\ +1/2$	$+3/2$	$3/2\ \ 1/2$
	1	$+1/2\ +1/2$

	$+1\ -1/2$	$0\ +1/2$	$3/2\ \ 1/2$
$+1\ \ 1/2$	$1/3$	$2/3$	$-1/2\ -1/2$
$0\ \ 1/2$	$2/3$	$-1/3$	$-1/2\ -1/2$

	$0\ -1/2$	$-1\ +1/2$	$3/2$
$0\ \ 1/2$	$2/3$	$1/3$	$-3/2$
$-1\ \ 1/2$	$1/3$	$-2/3$	$-3/2$

	$-1\ -1/2$	1
$-1\ \ 1/2$	1	1

$3/2\times1/2$

	2	
$+3/2\ +1/2$	$+2$	$2\quad 1$
	1	$+1\quad +1$

	$+3/2\ -1/2$	$+1/2\ +1/2$	$2\quad 1$
$+3/2\ \ 1/2$	$1/4$	$3/4$	$1\quad 0$
$+1/2\ \ 1/2$	$3/4$	$-1/4$	$1\quad 0$

	$+1/2\ -1/2$	$-1/2\ +1/2$	$2\quad 1$
$+1/2\ \ 1/2$	$1/2$	$1/2$	$1\quad 1$
$-1/2\ \ 1/2$	$1/2$	$-1/2$	$-1\quad -1$

	$-1/2\ -1/2$	$-3/2\ +1/2$	2
$+1/2\ \ 1/2$	$3/4$	$1/4$	2
$-1/2\ \ 1/2$	$1/4$	$-3/4$	-2

	$-3/2\ -1/2$	1
$-3/2\ \ 1/2$	1	1

2×1

	3	
$+2\ +1$	$+3$	$3\quad 2$
	1	$+2\quad +2$

	$+2\ \ 0$	$+1\ +1$	$3\quad 2\quad 1$
$+2\ +1$	$1/3$	$2/3$	$+1\quad +1\quad +1$
$+1\ +1$	$2/3$	$-1/3$	

	$+2\ -1$	$+1\ \ 0$	$0\ +1$	$3\quad 2\quad 1$
$+2\ +1$	$1/15$	$1/3$	$3/5$	$+1\quad +1\quad +1$
$+1\ +1$	$8/15$	$1/6$	$-3/10$	
$0\ +1$	$6/15$	$-1/2$	$1/10$	

	$+1\ -1$	$0\ \ 0$	$-1\ +1$	$3\quad 2\quad 1$
$+1\ +1$	$1/5$	$1/2$	$3/10$	$0\quad 0\quad 0$
$0\ \ 0$	$3/5$	0	$-2/5$	
$-1\ +1$	$1/5$	$-1/2$	$3/10$	

	$0\ -1$	$-1\ \ 0$	$-2\ +1$	$3\quad 2\quad 1$
$0\ +1$	$6/15$	$1/2$	$1/10$	$-1\quad -1\quad -1$
$-1\ +1$	$8/15$	$-1/6$	$-3/10$	
$-2\ +1$	$1/15$	$-1/3$	$3/5$	

	$-1\ -1$	$-2\ \ 0$	$-2\ -1$	$3\quad 2$
$-1\ +1$	$2/3$	$1/3$	3	$-2\quad -2$
$-2\ +1$	$1/3$	$-2/3$	-3	

	$-2\ -1$	1
$-2\ +1$	1	1

$3/2\times1$

	$5/2$	
$+3/2\ +1$	$+5/2$	$5/2\ \ 3/2$
	1	$+3/2\ +3/2$

	$+3/2\ \ 0$	$+1/2\ +1$	$5/2\ \ 3/2\ \ 1/2$
$+3/2\ +1$	$2/5$	$3/5$	$+1/2\ \ +1/2\ \ +1/2$
$+1/2\ +1$	$3/5$	$-2/5$	

	$+3/2\ -1$	$+1/2\ \ 0$	$-1/2\ +1$	$5/2\ \ 3/2\ \ 1/2$
$+3/2\ +1$	$1/10$	$2/5$	$1/2$	$-1/2\ \ -1/2\ \ -1/2$
$+1/2\ \ 0$	$3/5$	$1/15$	$-1/3$	
$-1/2\ +1$	$3/10$	$-8/15$	$1/6$	

	$+1/2\ -1$	$-1/2\ \ 0$	$-3/2\ +1$	$5/2\ \ 3/2$
$+1/2\ \ 1/2$	$3/10$	$8/15$	$1/6$	$-3/2\ \ -3/2$
$-1/2\ \ 1/2$	$3/5$	$-1/15$	$-1/3$	
$-3/2\ \ 1/2$	$1/10$	$-2/5$	$1/2$	

	$-1/2\ -1$	$-3/2\ \ 0$	$5/2\ \ 3/2$
$-1/2\ \ 1/2$	$3/5$	$2/5$	$-3/2\ \ -3/2$
$-3/2\ \ 1/2$	$2/5$	$-3/5$	

	$-3/2\ -1$	1
$-3/2\ \ 1/2$	1	1

1×1

	2	
$+1\ +1$	$+2$	$2\quad 1$
	1	$+1\quad +1$

	$+1\ \ 0$	$0\ +1$	$2\quad 1\quad 0$
$+1\ +1$	$1/2$	$1/2$	$+1\quad +1\quad +1$
$0\ +1$	$1/2$	$-1/2$	

	$+1\ -1$	$0\ \ 0$	$-1\ +1$	$2\quad 1\quad 0$
$+1\ +1$	$1/6$	$1/2$	$1/3$	$0\quad 0\quad 0$
$0\ \ 0$	$2/3$	0	$-1/3$	
$-1\ +1$	$1/6$	$-1/2$	$1/3$	

	$0\ -1$	$-1\ \ 0$	$-1\ -1$	$2\quad 1$
$0\ +1$	$1/2$	$1/2$	2	$-1\quad -1$
$-1\ +1$	$1/2$	$-1/2$	-2	

	$-1\ -1$	1
$-1\ +1$	1	1

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

$3/2\times3/2$

	3	
$+3/2\ +3/2$	$+3$	$3\quad 2$
	1	$+2\quad +2$

	$+3/2\ +1/2$	$+1/2\ +3/2$	$3\quad 2\quad 1$
$+3/2\ +1/2$	$1/2$	$1/2$	$+1\quad +1\quad +1$
$+1/2\ +3/2$	$1/2$	$-1/2$	

	$+3/2\ -1/2$	$+1/2\ +1/2$	$-1/2\ +3/2$	$3\quad 2\quad 1\quad 0$
$+3/2\ +1/2$	$1/5$	$1/2$	$3/10$	$0\quad 0\quad 0\quad 0$
$+1/2\ +1/2$	$3/5$	0	$-2/5$	
$-1/2\ +3/2$	$1/5$	$-1/2$	$3/10$	

	$+3/2\ -3/2$	$+1/2\ -1/2$	$-1/2\ +1/2$	$-3/2\ +3/2$	$3\quad 2\quad 1$
$+3/2\ +1/2$	$1/20$	$1/4$	$9/20$	$1/4$	$-1\quad -1\quad -1$
$+1/2\ +1/2$	$9/20$	$1/4$	$-1/20$	$-1/4$	
$-1/2\ +1/2$	$9/20$	$-1/4$	$-1/20$	$1/4$	
$-3/2\ +3/2$	$1/20$	$-1/4$	$9/20$	$-1/4$	

	$+1/2\ -3/2$	$-1/2\ -1/2$	$-3/2\ +1/2$	$3\quad 2$
$+1/2\ +1/2$	$1/5$	$1/2$	$3/10$	$-2\quad -2$
$-1/2\ +1/2$	$3/5$	0	$-2/5$	
$-3/2\ +1/2$	$1/5$	$-1/2$	$3/10$	

	$-1/2\ -3/2$	$-3/2\ -1/2$	$1/2\ \ 1/2$	3
$+1/2\ \ 1/2$	$1/2$	$1/2$	$1/2$	-3
$-3/2\ \ 1/2$	$1/2$	$-1/2$	$1/2$	-3

	$-3/2\ -3/2$	1
$-3/2\ \ 1/2$	1	1

$2\times3/2$

	$7/2$	
$+2\ +3/2$	$+7/2$	$7/2\ \ 5/2$
	1	$+5/2\ +5/2$

	$+2\ +1/2$	$+1\ +3/2$	$7/2\ \ 5/2\ \ 3/2$
$+2\ +3/2$	$3/7$	$4/7$	$+3/2\ \ +3/2\ \ +3/2$
$+1\ +3/2$	$4/7$	$-3/7$	

	$+2\ -1/2$	$+1\ \ 1/2$	$0\ \ 3/2$	$7/2\ \ 5/2\ \ 3/2\ \ 1/2$
$+2\ +1/2$	$1/7$	$16/35$	$2/5$	$+1/2\ \ +1/2\ \ +1/2\ \ +1/2$
$+1\ \ 1/2$	$4/7$	$1/35$	$-2/5$	
$0\ \ 3/2$	$2/7$	$-18/35$	$1/5$	

	$+2\ -3/2$	$+1\ -1/2$	$0\ \ 1/2$	$-1\ \ 3/2$	$2/5\quad 2/5$
$+2\ -1/2$	$1/35$	$6/35$	$2/5$		$-3/10$
$+1\ \ 1/2$	$12/35$	$5/14$	0		$1/5$
$0\ \ 1/2$	$18/35$	$-3/35$	$-1/5$		
$-1\ \ 3/2$	$4/35$	$-27/70$	$2/5$		$-1/10$

	$+3/2\ -3/2$	$+1/2\ -1/2$	$-1/2\ +1/2$	$-3/2\ +3/2$	$7/2\quad 5/2\quad 3/2\quad 1/2$
$+3/2\ +1/2$	$1/20$	$1/4$	$9/20$	$1/4$	$-1/2\ \ -1/2\ \ -1/2\ \ -1/2$
$+1/2\ +1/2$	$9/20$	$1/4$	$-1/20$	$-1/4$	
$-1/2\ +1/2$	$9/20$	$-1/4$	$-1/20$	$1/4$	
$-3/2\ +3/2$	$1/20$	$-1/4$	$9/20$	$-1/4$	

	$+1\ -3/2$	$0\ -1/2$	$-1\ \ 1/2$	$-2\ \ 3/2$	$7/2\quad 5/2\quad 3/2$
$+1\ +3/2$	$4/35$	$27/70$	$2/5$	$1/10$	$-3/2\ \ -3/2\ \ -3/2$
$0\ +1/2$	$18/35$	$3/35$	$-1/5$	$-1/5$	
$-1\ \ 1/2$	$12/35$	$-5/14$	0	$3/10$	
$-2\ \ 3/2$	$1/35$	$-6/35$	$2/5$	$-2/5$	

	$0\ -3/2$	$-1\ -1/2$	$-2\ \ 1/2$	$7/2\quad 5/2$
$0\ +1/2$	$2/7$	$18/35$	$1/5$	$-5/2\ \ -5/2$
$-1\ +1/2$	$4/7$	$-1/35$	$-2/5$	
$-2\ \ 1/2$	$1/7$	$-16/35$	$2/5$	

	$-1\ -3/2$	$-2\ -1/2$	$7/2$
$-1\ +1/2$	$4/7$	$3/7$	$7/2$
$-2\ +1/2$	$3/7$	$-4/7$	$-7/2$

	$-2\ -3/2$	1
$-2\ +1/2$	1	1

2×2

	4	
$+2\ +2$	4	$4\quad 3$
	1	$+3\quad +3$

	$+2\ +1$	$+1\ +2$	$4\quad 3\quad 2$
$+2\ +1$	$1/2$	$1/2$	$+2\quad +2\quad +2$
$+1\ +2$	$1/2$	$-1/2$	

	$+2\ \ 0$	$+1\ \ 1$	$0\ +2$	$4\quad 3\quad 2\quad 1$
$+2\ +2$	$3/14$	$1/2$	$2/7$	$+1\quad +1\quad +1\quad +1$
$+1\ +2$	$4/7$	0	$-3/7$	
$0\ +2$	$3/14$	$-1/2$	$2/7$	

	$+2\ -1$	$+1\ \ 0$	$0\ +1$	$-1\ +2$	$4\quad 3\quad 2\quad 1\quad 0$
$+2\ +1$	$1/14$	$3/10$	$3/7$	$1/5$	$0\quad 0\quad 0\quad 0\quad 0$
$+1\ \ 0$	$3/7$	$1/5$	$-1/14$	$-3/10$	
$0\ \ 1$	$3/7$	$-1/5$	$-1/14$	$3/10$	
$-1\ \ 2$	$1/14$	$-3/10$	$3/7$	$-1/5$	

	$+2\ -2$	$+1\ -1$	$0\ \ 0$	$-1\ \ 1$	$-2\ \ 2$	$4\quad 3\quad 2\quad 1$
$+2\ \ 0$	$1/70$	$1/10$	$2/7$	$2/5$	$1/5$	$-1\quad -1\quad -1\quad -1$
$+1\ \ 1$	$8/35$	$2/5$	$1/14$	$-1/10$	$-1/5$	
$0\ \ 2$	$18/35$	0	$-2/7$	0	$1/5$	
$-1\ \ 3$	$8/35$	$-2/5$	$1/14$	$1/10$	$-1/5$	
$-2\ \ 4$	$1/70$	$-1/10$	$2/7$	$-2/5$	$1/5$	

	$+1\ -2$	$0\ -1$	$-1\ \ 0$	$-2\ \ 1$	$4\quad 3\quad 2$
$+1\ +2$	$1/14$	$3/10$	$3/7$	$1/5$	$-2\quad -2\quad -2$
$0\ +2$	$3/7$	$1/5$	$-1/14$	$-3/10$	
$-1\ +2$	$3/7$	$-1/5$	$-1/14$	$3/10$	
$-2\ +2$	$1/14$	$-3/10$	$3/7$	$-1/5$	

	$0\ -2$	$-1\ -1$	$-2\ \ 0$	$4\quad 3$
$0\ +2$	$3/14$	$1/2$	$2/7$	$-3\quad -3$
$-1\ +2$	$4/7$	0	$-3/7$	
$-2\ +2$	$3/14$	$-1/2$	$2/7$	

	$-1\ -2$	$-2\ -1$	4
$-1\ +2$	$1/2$	$1/2$	4
$-2\ +2$	$1/2$	$-1/2$	-4

	$-2\ -2$	1
$-2\ +2$	1	1

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^1_{1,1} = \frac{1+\cos\theta}{2}$$

$$d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{3,2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2} \qquad d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta \qquad d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^{3,2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2} \qquad d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta \qquad d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta - 1}{2}\cos\frac{\theta}{2} \qquad d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2 \qquad d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1) \qquad d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta + 1}{2}\sin\frac{\theta}{2}$$

Figure 23.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The coefficients here have been calculated using computer programs written independently by Cohen and at LBNL.

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